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# Control of uncertain nonlinear systems with arbitrary relative degree and unknown control direction using sliding modes

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SUMMARY

The control of uncertain nonlinear systems by output feedback is addressed. A model-reference tracking sliding mode controller is designed for uncertain plants with arbitrary relative degree. Nonlinearities of a given class are incorporated as state dependent and possibly unmatched disturbances of a linear plant. Such class encompasses nonlinear systems which are triangular in the unmeasured states. In contrast with previous works, exact tracking is achieved by means of a switching strategy based on a locally exact differentiator, and a monitoring function is used to cope with the lack of knowledge of the control direction. Global or semi-global stability properties of the closed-loop system are proved. Copyright © 2007 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

In the adaptive control literature, the problem of controlling uncertain plants with unknown control direction, i.e. when the sign of the high-frequency gain is unknown, has been addressed since the early 1980s [1]. A solution to the problem appeared in [2] where the so-called Nussbaum gain was introduced to design stable adaptive control systems under this relaxed assumption. This concept

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became a standard design tool in adaptive control theory as in [3], and more recently in [4–6]. Although, in theory, this approach leads to a rigorous solution of the problem, it is of arguable practical interest, due to the large transients and lack of robustness that may result [3, 7].

In contrast to the adaptive control literature, few publications are available in the domain of sliding mode control (SMC) for this class of plants. In [8], SMC was proposed for a quite general class of uncertain nonlinear systems without the need of explicitly identifying the sign of the control direction. However, state feedback was required. In [9], a hybrid scheme was proposed for uncertain nonlinear systems with hard nonlinearities. Considering only first-order systems, it was argued that the proposed scheme could avoid the large transient resulting from the Nussbaum gain approach. An output feedback SMC scheme for tracking of uncertain linear plants with relative degree one was introduced in [10] where, in lieu of the Nussbaum gain, the controller was based on a switching algorithm driven by an appropriate monitoring function of the output error.

In this paper, we extend the controller of [10] to the case of nonlinear plants with arbitrary relative degree, using only output feedback. As in [11], the nonlinear plant is formulated as a linear plant with nonlinear disturbances which may be state dependent and unmatched. However, in [11], the control direction was assumed known and only practical asymptotic stability with small residual tracking error could be guaranteed. Here, we focus on the *exact* output tracking of uncertain nonlinear plants with *unknown* control direction.

The relative degree compensation and the asymptotic convergence of the tracking error to zero are achieved by means of a hybrid lead filter [12] which combines a conventional linear lead filter and a robust (locally) exact differentiator (RED) [13], based on 2-sliding modes.

To cope with the problem of unknown control direction, we propose a switching mechanism that adjusts the control sign through a monitoring function which depends on an appropriate auxiliary error. The new scheme is developed trying to retain the desirable qualities of the controller presented in [10] such as good transient performance and disturbance rejection capability. We also point out that our scheme seems applicable to plants with time-varying control direction, at least stepwise with sufficient time between steps.

In contrast with high gain observer-based schemes [14], no explicit state observers are employed and the control signal is free of peaking. Global or semi-global asymptotic stability with respect to a compact set is demonstrated. The proposed control strategy is evaluated by simulation. The applicability of the proposed controller in real-world conditions is supported by the experiments presented in [12, 15].

## 2. PROBLEM FORMULATION

*Notation*: The Euclidean norm of a vector $x$ and the corresponding induced norm of a matrix $A$ are denoted by $|x|$ and $|A|$, respectively. The $\mathscr{L}_{\infty e}$ norm of signal $x(t) \in \mathbb{R}^n$, from initial time $\bar{t}_0$, is defined as $\|x_{t,\bar{t}_0}\| := \sup_{\bar{t}_0 \leqslant \tau \leqslant t} |x(\tau)|$; for $\bar{t}_0 = 0$, $\|x_t\|$ is adopted. The symbol '$s$' represents either the Laplace variable or the differential operator '$\mathrm{d}/\mathrm{d}t$', according to the context. The output of a linear system with transfer function $H(s)$ and input $u$ is written $H(s)u$. Pure convolution $h(t) * u(t)$ is denoted by $H(s) * u$, with $h(t)$ being the impulse response of $H(s)$. Classes $\mathscr{K}$, $\mathscr{K}_\infty$ functions are defined as usual [14, p. 144]. ISS and ISpS mean input-to-state-stable (or stability) and input-to-state-practical-stability, respectively [16].



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Consider a single-input–single-output nonlinear uncertain plant described by

$$\dot{x} = f_p(x,t) + bu, \quad f_p(x,t) = Ax + \phi(x,t), \quad y = h^{\mathrm{T}}x \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the measured output and $\phi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is a state dependent uncertain nonlinear disturbance, possibly unmatched. The triple $(A, b, h^{\mathrm{T}})$ is in the canonical controllable form with uncertain constant matrices $A$ (lower companion form) and $h^{\mathrm{T}} = [h_1 \; h_2 \; \dots \; h_{(n-n^*+1)} \; 0 \; \dots \; 0]$. Note that $h_{(n-n^*+1)} = h^{\mathrm{T}} A^{n^*-1} b$, which complies with the general expression for the high frequency gain (HFG) of the linear subsystem $(A, b, h^{\mathrm{T}})$ [14, p. 512].

### 2.1. *Basic assumptions*

Without loss of generality, we assume that the initial time is $t = 0$. All uncertain parameters belong to some compact set $\Omega_p$ such that the necessary uncertainty bounds to be defined later are available for design. In $\Omega_p$, we assume that: (i) $\phi$ is locally Lipschitz in $x$ $(\forall x)$, and sufficiently smooth in its arguments; (ii) $(A, b, h^{\mathrm{T}})$ represents a linear plant which is minimum phase, observable, has known order $n$ and known *relative degree* $n^*$, as usual in model reference adaptive control (MRAC) [17].

Filippov's definition for the solution of discontinuous differential equations is assumed [18] throughout the paper. In order to avoid clutter, the symbol $u$ alone, without the argument $t$, represents a switching control law which is not a usual function of $t$ when sliding mode takes place. On the other hand, we denote the extended equivalent control [19] and [20, Section 2.3] by $u(t)$ (instead of $u_{\mathrm{eq}}(t)$) which, by definition, is piecewise continuous. Note that $u$ can always be replaced by $u(t)$ in the right-hand side of the governing differential equations. Our main additional assumptions are:

(I) There exists a global diffeomorphism $(\bar{x}, t) = T(x, t)$, $\bar{x}^{\mathrm{T}} := [\eta^{\mathrm{T}} \; \xi^{\mathrm{T}}]$, $\eta \in \mathbb{R}^{n-n^*}$, which transforms (1) into the *normal form* [14, p. 516], with $\xi = [y \; \dot{y} \dots y^{(n^*-1)}]^{\mathrm{T}}$ and

$$\dot{\eta} = F_0(\eta, \xi), \quad \dot{\xi} = A_r \xi + B_r k_p [u + d(x,t)], \quad y = \xi_1$$

where $k_p := h^{\mathrm{T}} A^{n^*-1} b = h_{(n-n^*+1)}$ is the *constant* plant HFG, $(A_r, B_r)$ is in the Brunovsky's controller form and the $\eta$-dynamics is ISS from $\xi$ to $\eta$.

According to (I), the plant (1) has uniform relative degree $n^*$ and the HFG $k_p$ is constant. Here, the classical assumption about the *prior* knowledge of the control direction is removed, i.e. $k_p$ is uncertain in *norm and sign*.

The above assumption is satisfied by systems (1) with $\phi$ triangular in the unmeasured states $\phi(x,t) = [\phi_1(x_1, y, t) \; \phi_2(x_1, x_2, y, t) \dots \phi_n(x_1, \dots, x_n, y, t)]^{\mathrm{T}}$. It would be desirable to characterize more general systems that satisfy such assumption. We further assume that:

(II) The term $\phi$ is norm bounded by $|\phi(x,t)| \leqslant k_x |x| + \varphi(y,t)$, $\forall x, t$, where $k_x \geqslant 0$ is a *known* scalar and $\varphi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a known function piecewise continuous in $t$ and continuous in $y$, and $\varphi(y,t) \leqslant \Psi_\varphi(|y|) + k_\varphi$, where $\Psi_\varphi \in \mathscr{K}_\infty$ is locally Lipschitz and $k_\varphi > 0$ is a constant.

According to (II), no particular growth condition, such as linear growth or existence of a global Lipschitz constant, is imposed on $\varphi$. Therefore, nonlinearities like $\varphi(y) = y^2$ can be included. Then, finite-time escape is not precluded *a priori* and for each solution of (1) there exists a maximal time interval of definition given by $[0, t_M)$, where $t_M$ may be finite or infinite.



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### 2.2. Control objective

The aim is to achieve global or semi-global stability properties in the sense of uniform signal boundedness and asymptotic output tracking, i.e. the *output tracking error*

$$e(t)=y(t)-y_m(t) \tag{2}$$

should asymptotically tend to zero (exact tracking). The *desired trajectory* $y_m(t)$ is assumed to be generated by the following *reference model*:

$$y_m=M(s)r=\frac{k_m}{L(s)(s+a_m)}r, \quad k_m, a_m>0, \quad L(s)=s^{(n^*-1)}+\sum_{i=2}^{n^*} l_{n^*-i}s^{(n^*-i)} \tag{3}$$

where the *reference signal* $r(t)$ is assumed piecewise continuous and uniformly bounded and $L(s)$ is a Hurwitz polynomial.

## 3. OUTPUT TRACKING ERROR EQUATION

In this section, an output feedback model matching control $u^*$ is derived so that, when $u=u^*$, the transfer function of the closed-loop system is the same as that of the model. Then, the relevant output error equation is obtained. To this end, a key idea is to transform $\phi$ to an input (matched) disturbance.

### 3.1. Output feedback model matching control

In order to obtain an output feedback model matching control, we first introduce the *regressor vector* $\omega:=[\omega_1^{\mathrm{T}}\ \omega_2^{\mathrm{T}}\ y\ r]^{\mathrm{T}}$, using the following input and output filters of MRAC design [17]:

$$\dot{\omega}_1=\Lambda\omega_1+gu, \quad \dot{\omega}_2=\Lambda\omega_2+gy \tag{4}$$

where $\Lambda\in\mathbb{R}^{(n-1)\times(n-1)}$ is Hurwitz and $g$ is a constant vector such that $(\Lambda, g)$ is controllable. Such filters are needed due to the lack of full state measurement of the plant and replace a state observer. Then, the model matching control is parametrized as

$$u=\theta^{\mathrm{T}}\omega, \quad \theta^{\mathrm{T}}:=[\theta_1^{\mathrm{T}}\ \theta_2^{\mathrm{T}}\ \theta_3\ \theta_4] \tag{5}$$

If $\phi\equiv 0$, the closed-loop transfer function from $r$ to $y$ is denoted by $G_{\mathrm{c}}(s,\theta)$. As is well known [17], there exists a constant vector $\theta^*$ which solves the equation $G_{\mathrm{c}}(s,\theta)=M(s)$ provided that the zeros of the model are eigenvalues of $\Lambda$. Thus, if $\phi\equiv 0$, a *model matching control law* is given by $u^*=\theta^{*\mathrm{T}}\omega$. Further, $\theta^*$ is unique if the model is of order $n$. In particular, model matching requires $\theta_4^*=k_m/k_p$. Since the plant parameters are uncertain, $\theta^*$ is not available. However, we assume that $\theta^*$ is elementwise norm bounded by a known constant vector $\bar{\theta}$. Thus, $u^*$ can also be norm bounded with available signals.

### 3.2. Error equation and equivalent nonlinear input disturbance

With $X^{\mathrm{T}}:=[x^{\mathrm{T}}\ \omega_1^{\mathrm{T}}\ \omega_2^{\mathrm{T}}]$, $u$ replaced by $u-u^*+u^*$, and noting that, for appropriate matrices $\Omega_1$ and $\Omega_2$, $\omega=\Omega_1X+\Omega_2r$, one can write the state-space representation of (1) and (4) as

$$\dot{X}=A_{\mathrm{c}}X+b_{\mathrm{c}}r+b_{\mathrm{c}}k^*[u-u^*]+B_\phi\phi, \quad y=h_{\mathrm{c}}^{\mathrm{T}}X \tag{6}$$

where $B_\phi^{\mathrm{T}} = [I \ 0 \ 0]$ and $k^* := 1/\theta_4^*$. Note that $(A_{\mathrm{c}}, b_{\mathrm{c}}, h_{\mathrm{c}}^{\mathrm{T}})$ is a non-minimal stable realization of $M(s)$. The desired trajectory $y_m$ can also be generated by

$$\dot{X}_m = A_{\mathrm{c}} X_m + b_{\mathrm{c}} k^* [\theta_4^* r - d_\phi] + B_\phi \phi, \quad y_m = h_{\mathrm{c}}^{\mathrm{T}} X_m \tag{7}$$

where the *equivalent input disturbance* $d_\phi = (k^* M(s))^{-1} h_{\mathrm{c}}^{\mathrm{T}} (sI - A_{\mathrm{c}})^{-1} B_\phi \phi$ can be written as

$$d_\phi := W_{n^*-1} \phi^{(n^*-1)} + \cdots + W_1 \dot{\phi} + W_0 \phi + \bar{W}_\phi(s) * \phi \tag{8}$$

with $\bar{W}_\phi$ being a row vector of strictly proper and BIBO stable transfer functions and $W_i \in \mathbb{R}^n$ are constant row vectors obtained from the model parameters and the Markov parameters corresponding to $h_{\mathrm{c}}^{\mathrm{T}}(sI - A_{\mathrm{c}})^{-1} B_\phi$. Note that from the relative degree assumption of (I), $u$ does not appear in $d_\phi$ which involves the derivatives of the output $y$ only up to order $n^* - 1$. Now, from (6) and (7), one has

$$\dot{X}_e = A_{\mathrm{c}} X_e + b_{\mathrm{c}} k^* [u - \bar{u}], \quad e = h_{\mathrm{c}}^{\mathrm{T}} X_e, \quad X_e := X - X_m \tag{9}$$

$$e = k^* M(s)[u - \bar{u}], \quad \bar{u}(t) := \theta^{*\mathrm{T}} \omega - d_\phi \tag{10}$$

where $\bar{u}$ is the model matching control in the presence of $\phi$ [11].

## 4. NORM BOUND FOR EQUIVALENT DISTURBANCE

Since we assume sufficient differentiability for $\phi$, one can find $\Psi_\phi \in \mathscr{K}$ and a constant $k_\phi > 0$ such that $|d_\phi| \leqslant \Psi_\phi(|x|) + \bar{W}_\phi(s) * |\phi| + k_\phi$. Considering (II) and applying [21, Lemma 3] to (6), it is possible to find $k_x^* > 0$ such that, for $k_x \in [0, k_x^*]$ a norm bound for $X$ and $x$ can be obtained through *first-order approximation filters* (FOAFs) (see details in [21]). Therefore, one has $|x(t)| \leqslant \hat{x}(t) + \hat{\pi}(t)$, where

$$\hat{x}(t) := \frac{1}{s + \lambda_x} [c_1 \varphi(y, t) + c_2 |\omega(t)|] \tag{11}$$

with $c_1, c_2, \lambda_x > 0$ being appropriate constants. The exponentially decaying term $\hat{\pi}$ accounts for initial conditions [21]. Moreover, from (II) and (11), one has $|\phi(x, t)| \leqslant k_x \hat{x}(t) + \varphi(y, t)$, *modulo* $\hat{\pi}$ term and one can write $|d_\phi| \leqslant \hat{d}_\phi + \hat{\pi}_\phi$, where $\hat{\pi}_\phi$ is a decaying term,

$$\hat{d}_\phi(t) := \Psi_\phi(|\hat{x}(t)|) + \frac{c_\phi}{s + \gamma_\phi} [k_x \hat{x}(t) + \varphi(y, t)] + k_\phi \tag{12}$$

and $c_\phi/(s + \gamma_\phi)$ is a FOAF designed for $\bar{W}_\phi(s)$, with appropriate constants $c_\phi, \gamma_\phi > 0$.

## 5. OUTPUT FEEDBACK SLIDING MODE CONTROLLER FOR PLANTS WITH KNOWN CONTROL DIRECTION

For plants with $n^* = 1$, $M(s)$ in (3) is strictly positive real (SPR). Applying [22, Lemma 1] to the error equation (10), global exponential stability and finite time exact tracking are guaranteed



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with $u=-[\mathrm{sgn}(k_p)]f(t)\,\mathrm{sgn}(e)$ where the *modulation function* $f(t)$ satisfies $f(t)\geqslant|\bar{u}|+\delta$, with $\bar{u}$ defined in (10) and $\delta>0$ being an arbitrarily small constant. For the case of plants with $n^*>1$, $M(s)$ is not SPR. However, with the multiplier $L(s)$, $ML(s)$ is SPR and from (10):

$$\bar{e}=k^*ML[u-\bar{u}] \quad \text{with} \quad \bar{e}=L(s)e=\mathrm{e}^{(n^*-1)}+l_{n^*-2}\mathrm{e}^{(n^*-2)}+\cdots+l_1\dot{e}+l_0e \tag{13}$$

Thus, using $u=-[\mathrm{sgn}(k_p)]f(t)\,\mathrm{sgn}(\bar{e})$ we recover the $n^*=1$ case results. The problem is that the *ideal sliding variable* $\bar{e}$ is not available since $L(s)$ is non-causal. This motivates the following relative degree compensation strategy.

## 6. RELATIVE DEGREE COMPENSATION

As in [15, 23], a *hybrid lead filter* (Figure 1) introduced in [12], named global robust exact differentiator (GRED), will be used to estimate $\bar{e}$. The GRED provides a surrogate for the non-causal operator $L(s)$ (3) by combining a linear lead filter with a 2-sliding mode-based RED, by means of a suitable switching scheme.

### 6.1. Linear lead filter

The linear lead filter is given by

$$\hat{e}_l=L_a(s)e, \quad L_a(s)=L(s)/F(\tau s), \quad F(\tau s)=(\tau s+1)^{(n^*-1)}, \quad \tau>0 \tag{14}$$

As $\tau$ tends to zero, $L_a(s)$ and $\hat{e}_l$ approximate $L(s)$ and $\bar{e}$, respectively. Henceforth, let $\tau\in(0,\bar{\tau}]$, where $\bar{\tau}<1$ is some sufficiently small constant. When $L_a(s)$ is used in a SMC loop, global/semi-global stability properties can be guaranteed, even in the presence of an additive disturbance $\beta_\alpha$

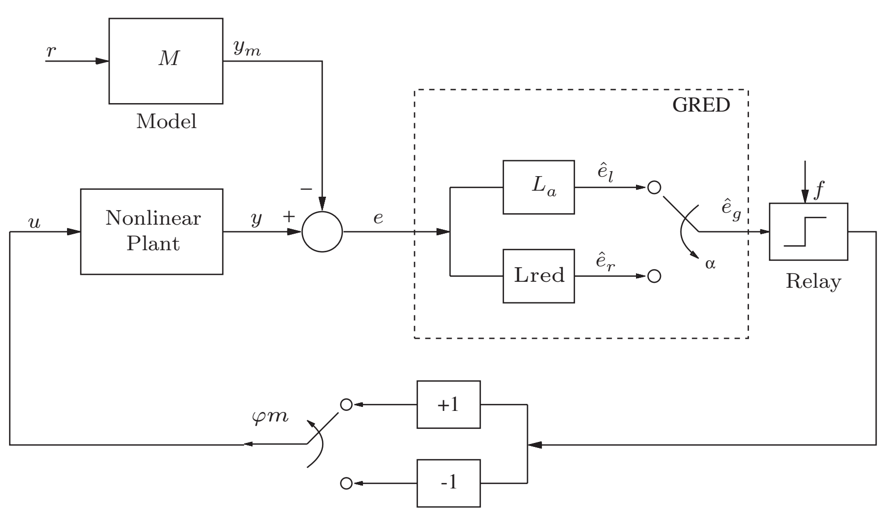


Figure 1. Output sliding mode controller with a hybrid lead filter (GRED) for relative degree compensation and a monitoring scheme ($\varphi_m$) to adjust the control sign.



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of order $\mathcal{O}(\tau)$ in $\hat{e}_l$ [12]. However, the linear lead filter cannot provide the exact estimate of $\bar{e}$, and is well known to lead to control chattering with residual tracking error. Alternatively, one could think of using the so-called 2-sliding exact differentiators to achieve exact tracking.

### 6.2. *Robust exact differentiator*

The following differentiator, proposed in [13], is used:

$$
\begin{aligned}
\dot{\eta}_0 &= v_0, \quad v_0 = -\lambda_0 |\eta_0 - e(t)|^{n/(n+1)} \operatorname{sgn}(\eta_0 - e(t)) + \eta_1 \\
&\vdots \\
\dot{\eta}_i &= v_i, \quad v_i = -\lambda_i |\eta_i - v_{i-1}|^{(n-i)/(n-i+1)} \operatorname{sgn}(\eta_i - v_{i-1}) + \eta_{i+1} \\
&\vdots \\
\dot{\eta}_n &= -\lambda_n \operatorname{sgn}(\eta_n - v_{n-1})
\end{aligned}
\tag{15}
$$

According to [13, Theorem 1], if the parameters $\lambda_i$ $(i = 0, \ldots, n)$ are properly chosen, (15) can provide the exact derivatives, in the absence of noise, after a finite time transient process. Hence, $\bar{e}$ (13) can be estimated by the signal

$$
\hat{e}_r = \eta_{n^*-1} + l_{n^*-2}\eta_{n^*-2} + \cdots + l_1\eta_1 + l_0\eta_0 \tag{16}
$$

and the nonlinear lead compensator ($L_{\text{red}}$), defined by the RED (15) with order $n^* - 1$, input $e$ and output $\hat{e}_r$, is an alternative approximation for $L(s)$.

Even though the RED can provide the exact estimate of $\bar{e}$, when used in the feedback loop only local convergence can be guaranteed, since the signal we need to differentiate is the output error and a Lipschitz constant holds only locally for this signal [13]. In the following, a new estimate for $\bar{e}$ is obtained, trying to retain the desirable features of both estimates $\hat{e}_l$ and $\hat{e}_r$.

### 6.3. *Global robust exact differentiator*

The GRED (see Figure 1) is a block composed by $L_a$ and $L_{\text{red}}$ with output $\hat{e}_g$ formed by the following convex combination:

$$
\hat{e}_g = \alpha(\tilde{e}_{rl})\hat{e}_l + [1 - \alpha(\tilde{e}_{rl})]\hat{e}_r, \quad \tilde{e}_{rl} = \hat{e}_r - \hat{e}_l \tag{17}
$$

where $\hat{e}_l$ (14) and $\hat{e}_r$ (16) are the estimates of $\bar{e}$ provided by $L_a$ and $L_{\text{red}}$, respectively. The continuous switching function $\alpha : \mathbb{R} \to [0, 1]$ allows smooth changes between $\hat{e}_l$ and $\hat{e}_r$:

$$
\alpha(\tilde{e}_{rl}) = \begin{cases} 0 & \text{for } |\tilde{e}_{rl}| < \varepsilon_M - c \\ (|\tilde{e}_{rl}| - \varepsilon_M + c)/c & \text{for } \varepsilon_M - c \leqslant |\tilde{e}_{rl}| < \varepsilon_M \\ 1 & \text{for } |\tilde{e}_{rl}| \geqslant \varepsilon_M \end{cases} \tag{18}
$$

where $0 < c < \varepsilon_M$ and $\varepsilon_M := \tau K_R$, with $K_R$ being an appropriate positive design parameter and $\tau$ in (14). For high values of $|\tilde{e}_{rl}|$, the estimate $\hat{e}_l$ is chosen to ensure closed-loop stability, as explained below (14), while for small values of $|\tilde{e}_{rl}|$ we can choose the estimate $\hat{e}_r$ to guarantee ultimate exact estimation [12].



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From (17), it can be concluded that $\beta_\alpha := \hat{e}_g - \hat{e}_l = (1-\alpha)\tilde{e}_{rl}$ and, with (18), one has

$$\hat{e}_g = \hat{e}_l + \beta_\alpha \quad \text{and} \quad |\beta_\alpha| \leqslant \varepsilon_M \ (= \tau K_R) \tag{19}$$

which means that the resulting system is equivalent to a SMC using a lead filter compensation in the presence of the output disturbance $\beta_\alpha$, which is uniformly bounded and of order $\mathcal{O}(\tau)$. Thus, global or semi-global stability properties of the overall closed-loop system can be assured and ultimately exact estimation of $\bar{e}$ can be obtained [12].

## 7. PEAKING PHENOMENON

The output $\hat{e}_l(t)$ of (14) will contain a transient term of the form $(a/\tau^b)\mathrm{e}^{-ct/\tau}$, for some $a, b, c > 0$. Thus, due to $\hat{e}_l(t)$, $\hat{e}_g(t)$ eventually exhibits an impulsive-like transient behaviour, as $\tau \to 0$, where the transient peaks to $\mathcal{O}(1/\tau)$ values before it decays rapidly to zero. As in high-gain observer-based schemes, this behaviour is known as the *peaking phenomenon* [14].

However, as in [24], the peaking phenomenon can be circumvented by using the *peak extinction time* ($t_e$) concept, where $t_e$ is defined as the solution of $(a/\tau^b)\mathrm{e}^{-ct_e/\tau} = 1$, for each value of $\tau \in (0, 1]$. Note that $t_e$ is a function of $\tau$, which satisfies $t_e(\tau) \leqslant \bar{t}_e(\tau) (\in \mathcal{K})$ [24].

Applying the state variable transformation $T := \mathrm{diag}\{1/\tau^{(n^*-1)}, 1/\tau^{(n^*-2)}, \ldots, 1/\tau\}$ to the canonical controllable realization of $\tau^{(n^*-1)} L_a(s)$, one gets the following realization for (14):

$$\tau \dot{x}_f = A_f x_f + B_f e, \quad \tau^{(n^*-1)} \hat{e}_l = C_f(\tau) x_f + e \tag{20}$$

where $A_f$ and $B_f$ are constants matrices independent of $\tau$. Changing the timescale in (20) with $t = \tau\bar{t}$, the first equation in (20) becomes independent of $\tau$. Therefore, the state $x_f$ does not have peaking. Since $C_f(\tau)$ is finite as $\tau \to 0$, then peaking appears only at $\hat{e}_l$.

## 8. OUTPUT FEEDBACK SLIDING MODE CONTROLLER FOR PLANTS WITH UNKNOWN CONTROL DIRECTION

In [10], a monitoring function was developed to cope with the lack of knowledge of the control direction. Only linear plants with $n^* = 1$ were considered and the ideal sliding variable, i.e. the output tracking error, was chosen as the monitored variable. Here, for plants with $n^* > 1$, a monitoring function $\varphi_m$ for the ideal sliding variable estimate $\hat{e}_g$ (17) is used to decide when the control signal (see Figure 1)

$$u = \begin{cases} u^+ = -f(t)\,\mathrm{sgn}(\hat{e}_g), & t \in T^+ \\ u^- = f(t)\,\mathrm{sgn}(\hat{e}_g), & t \in T^- \end{cases} \tag{21}$$

should be switched from $u^+$ to $u^-$ and vice versa. In (21), $T^+ \cup T^- = [0, t_M)$, $T^+ \cap T^- = \emptyset$ and both $T^+$ and $T^-$ have the form $[t_k, t_{k+1}) \cup \cdots \cup [t_l, t_{l+1})$, where $t_k$ or $t_l$ denote switching times. A possible choice for a modulation function $f(t)$ to satisfy the inequality $f(t) \geqslant |\bar{u}| + \delta$ (see Section 5), *modulo* exponentially decaying terms, is given by

$$f(t) = |\bar{\theta}||\omega(t)| + |\hat{d}_\phi(t)| + \delta \tag{22}$$



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with $\hat{d}_\phi$ in (12) and an arbitrary constant $\delta>0$. Thus, $f(t)$ can be implemented using only available signals. Moreover, the $\operatorname{sgn}(\cdot)$ function in (21) blocks the transmission of an eventual peaking (due to $\hat{e}_l$) to the plant, assuring a peaking-free control signal. The parameter vector $\bar{\theta}^{\mathrm{T}}$ is such that $\bar{\theta}_i > \max\{|\theta_i^*|, |\theta_i^\dagger|\}$, where $\theta^\dagger$ is the model matching vector w.r.t. an *unstable* reference model $M^\dagger = k_m/L(s)(s-a_m)$ , $k_m, a_m>0$. This will guarantee that, with wrong control direction, the system would become unstable and consequently $\operatorname{sgn}(k_p)$ could be correctly found with $\varphi_m$.

## 9. MONITORING FUNCTION

We now construct the monitoring function $\varphi_m$ based on a norm bound for $\hat{e}_g$ developed in what follows. From (10) and (13), $\hat{e}_l$ (14) can be rewritten as

$$\hat{e}_l = k^* ML(s)[u-\bar{u}] + \beta_l + e_F^0, \quad \beta_l := k^* ML(s)[1-F(\tau s)]F^{-1}(\tau s) * (u-\bar{u}) \tag{23}$$

where the initial conditions of $X_e$ in (9) and $x_f$ in (20) are incorporated in the term

$$e_F^0 := L_a(s) * h_c^{\mathrm{T}} \mathrm{e}^{A_c t} X_e(0) - h_L^{\mathrm{T}} \mathrm{e}^{A_c t} X_e(0) + \frac{1}{\tau^{(n^*-1)}} C_f \mathrm{e}^{A_f t/\tau} x_f(0) \tag{24}$$

where $h_L^{\mathrm{T}} = h_c^{\mathrm{T}} A_c^{n^*-1} + \sum_{i=0}^{n^*-2} h_c^{\mathrm{T}} A_c^i l_i$. Using (19) it is possible to verify that $\hat{e}_g$ satisfies

$$\hat{e}_g = k^* ML(s)[u-\bar{u}] + \beta + e_F^0, \quad \beta = \beta_l + \beta_\alpha \tag{25}$$

From (23) and $|u-\bar{u}| \leqslant 2f$, it follows that $|\beta_l| \leqslant 2W_\beta(s,\tau) * f(t)$, where $W_\beta(s,\tau)$ can be chosen, through partial fraction expansion of $k^* ML(s)[1-F(\tau s)]F^{-1}(\tau s)$, as a sum of two FOAFs, one of them with a fast pole $-1/\tau$ and such that the induced $\mathscr{L}_\infty$ norm of the operator $W_\beta(s,\tau)$ is of order $\mathcal{O}(\tau)$. Since $|\beta_\alpha| \leqslant \varepsilon_M$ (see (19)), the unmeasurable signal $\beta$ can be norm bounded by the available signal

$$\bar{\beta} = 2W_\beta(s,\tau) * f(t) + \varepsilon_M \tag{26}$$

### 9.1. *Upper bound for the estimate $\hat{e}_g$*

From (24) and the partial fraction expansion of $L_a(s)$ in (14), one has

$$|e_F^0| \leqslant R_1 \mathrm{e}^{-\lambda_c t} + \frac{R_2}{\tau^{(n^*-1)}} \mathrm{e}^{-t/\tau} \quad \forall t \in [0, t_M) \tag{27}$$

where $R_1$ and $R_2$ are linear combinations of $|X_e(0)|$ and $|x_f(0)|$ (independent of $\tau$) and $0<\lambda_c< \min_i\{-Re(\lambda_i[A_c])\}$, with $\lambda_i[A_c]$ being the spectrum of $A_c$ in (9). Moreover, from (27) one can verify that

$$|e_F^0| \leqslant R_a \mathrm{e}^{-\lambda_a(t-\bar{t}_e(\tau))} \quad \forall t \in [\bar{t}_e, t_M), \quad R_a = k_a(|X_e(0)| + |x_f(0)|) \tag{28}$$

where $0<\lambda_a<\min(\lambda_c, 1/\bar{\tau})$, $\tau<\bar{\tau}<1$, $k_a>0$ is a constant and $\bar{t}_e(\tau)$ is an upper bound for the *peak extinction time* $t_e$, defined in Section 7, which can be obtained from the known upper bounds of the plant parameters [24]. Now, consider the following function:

$$\gamma(t) := R_a \mathrm{e}^{-\lambda_a(t-\bar{t}_e)} + f_d(t), \quad f_d(t) := \begin{cases} \|\bar{\beta}_{t,\bar{T}_j}\|, & \bar{T}_j < t \leqslant T_{j+1} \\ \|\bar{\beta}_{t,\bar{T}_{j-1}}\| \mathrm{e}^{-\sigma(t-T_j)}, & T_j < t \leqslant \bar{T}_j \end{cases} \tag{29}$$



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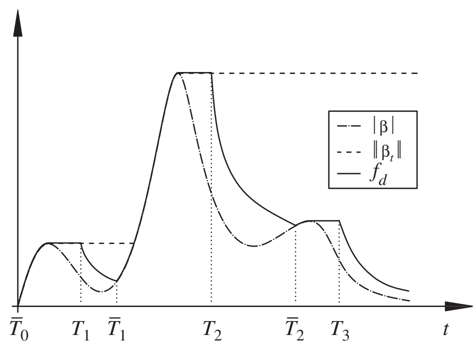


Figure 2. Functions $|\beta|$, $\|\beta_t\|$ and $f_d$.

where $j=\{0,1,2,\ldots\}$ and $\sigma\leqslant a_m$ with $a_m$ of (3). Moreover (see Figure 2),

$$T_{j+1}:=\min\{t>\bar{T}_j:|\bar{\beta}(t)|\leqslant\mu f_d(t)\},\quad \bar{T}_j:=\min\{t>T_j:f_d(t)\leqslant|\bar{\beta}(t)|\}\tag{30}$$

where $0<\mu<1$ and, by convention, $\bar{T}_0=0$. If *the control direction is correctly estimated*, then, applying *Lemma 1* (see Appendix A) to (25), a norm bound for $\hat{e}_g$ can be obtained. Indeed, choosing (29) as the $\gamma$-function of *Lemma 1*, the following upper bound is valid $\forall t\in[t_i,t_M)$, with $\bar{t}_e\leqslant t_i\leqslant t<t_M$:

$$|\bar{e}(t)|,|\hat{e}_g(t)|\leqslant\zeta(t),\quad \zeta(t):=(|\hat{e}_g(t_i)|+|\bar{\beta}(t_i)|)\mathrm{e}^{-a_m(t-t_i)}+(2R_a\mathrm{e}^{\lambda_a\bar{t}_e})\mathrm{e}^{-\lambda_a t}+2f_d(t)\tag{31}$$

We will construct a monitoring function $\varphi_m$ based on the upper bound (31). The decaying rate $\sigma$ acts like a *forgetting factor* which provides a vanishing property for $f_d$ in (29), see Figure 2. This allows a less conservative monitoring function which results in better transient response.

### 9.2. Implementation of the monitoring function

Now, consider the function

$$\varphi_k(t):=(|\hat{e}_g(t_k)|+|\bar{\beta}(t_k)|)\mathrm{e}^{-a_m(t-t_k)}+a(k)\mathrm{e}^{-\lambda_c t}+2f_d(t)\quad \forall t\in[t_k,t_{k+1})\tag{32}$$

where $a(k)$ is any positive monotonically increasing unbounded sequence. The motivation behind the introduction of $a(k)\mathrm{e}^{-\lambda_c t}$ is that the term $(2R_a\mathrm{e}^{\bar{\lambda}_a\bar{t}_e})\mathrm{e}^{-\bar{\lambda}_a t}$ in (31) is not available for measurement. According to [17, p. 340], the eigenvalues of $A_c$ are the zeros of the plant, say $z_k$, the poles of the reference model and the poles of the input/output filters which generate the regressor vector. Thus $\lambda_c$ satisfying the inequality below (27) can be found provided that a lower bound for $-Re(z_k)$ is known. The *monitoring function* $\varphi_m$ can thus be defined as

$$\varphi_m(t):=\varphi_k(t)\quad \forall t\in[t_k,t_{k+1})\subset[0,t_M)\tag{33}$$

Note that from (32) and (33), one always has $|\hat{e}_g(t_k)|<\varphi_k(t_k)$ at $t=t_k$. Hence, the switching time $t_k$ from $u^-$ to $u^+$ (or $u^+$ to $u^-$) is defined by

$$t_{k+1}:=\begin{cases}\min\{t>t_k:|\hat{e}_g(t)|=\varphi_k(t)\} & \text{if it exists}\\ t_M & \text{otherwise}\end{cases}\tag{34}$$

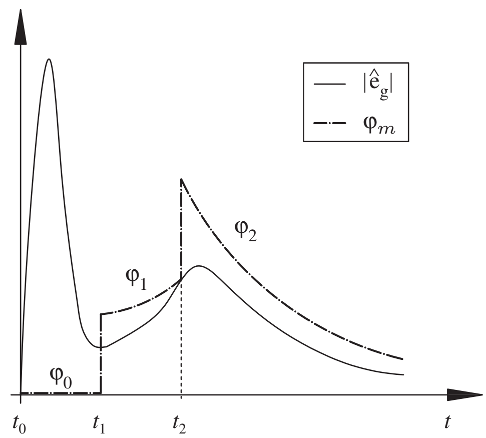


Figure 3. The trajectories of $\varphi_m$ and $|\hat{e}_g|$.

where $k \in \{1, 2, \ldots\}$, $t_0 := 0$ and $t_1 := \bar{t}_e$. For convenience, $\varphi_0 := 0$, $\forall t \in [t_0, t_1)$, see Figure 3. The following inequality is directly obtained from definition (33):

$$|\hat{e}_g(t)| \leqslant \varphi_m(t) \quad \forall t \in [t_1, t_M) \tag{35}$$

Figure 3 illustrates the estimate $|\hat{e}_g|$ with initial peaking as well as the monitoring function $\varphi_m$.

## 10. STABILITY RESULT

In order to fully account for the initial conditions involved in (9) and (25), let

$$z^{\mathrm{T}} := [X_e^{\mathrm{T}}, x_f, (z^0)^{\mathrm{T}}] \tag{36}$$

where $z^0$ denotes the *transient state* [22] corresponding to the BIBO stable filters used in (22). The main stability and convergence result is now stated.

*Theorem 1*
Assume that (I)–(II) and (22) hold. Then, for sufficiently small $\tau > 0$, the complete error system (9), (21) and (25), with state $z(t)$, is globally/semi-globally asymptotically stable w.r.t. a compact set independent of the initial conditions. Moreover, the ideal sliding mode $\bar{e} \equiv 0$ is achieved in finite time, thus avoiding chattering. The error signals $z(t)$ and $e(t)$ tend exponentially to zero and the control sign switching stops at the correct sign.

*Proof*
See Appendix A. □

## 11. SIMULATION RESULTS

To illustrate the performance of the proposed controller, consider the nonlinear plant (1) with $(A, b, h^{\mathrm{T}})$ being the controllable canonical realization of the unstable transfer function



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Table I. Control system (simulation results).

| Element | Value |
|---|---|
| Reference model | $M(s) = \dfrac{4}{(s+2)^3}, \quad r(t) = 5\sin(t)$ |
| FOAFs for $\bar{\beta}$ (26) | $\bar{k}^* = 1, \quad W_\beta(s,\tau) = \dfrac{4\tau}{s+2} + \dfrac{5.2}{s+1/\tau}$ |
| Monitoring function (32) and (33) | $a(k) = k+1, \quad a_m = 2, \quad \lambda_c = 1, \quad t_1 = \bar{t}_e = 0.1\,\mathrm{s}$ |
| Function $f_d$ (29) and (30) | $\sigma = 1, \quad \mu = 0.8$ |
| *Lead filter* | $L(s) = (s+2)^2, \quad F(\tau s) = (\tau s + 1)^2, \quad \tau = 10^{-3}$ |
| RED parameters | $\lambda_0 = 3C_3^{1/3}, \quad \lambda_1 = 1.5C_3^{1/2}, \quad \lambda_2 = 1.1C_3, \quad C_3 = 250$ |
| Switching law $\alpha$ (18) | $\varepsilon_M = 600\tau, \quad c = 10\tau$ |

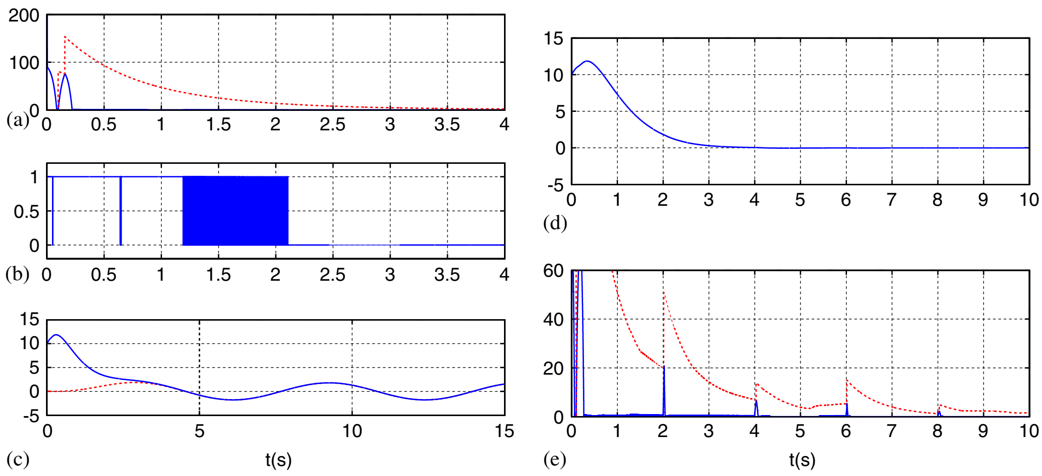


Figure 4. Simulation results: (a) auxiliary error $|\hat{e}_g|$ (—) and monitoring function $\varphi_m$ (– –); (b) switching law $\alpha$; (c) plant output $y$ (—) and reference model output $y_m$ (– –); (d) output error $e$; and (e) auxiliary error $|\hat{e}_g|$ (—) and monitoring function $\varphi_m$ (– –).

$G(s) = 1/(s+2)(s+1)(s-1)$ with relative degree $n^* = 3$ and $\phi^{\mathrm{T}}(x,t) = [x_1^2,\ 0,\ x_2 \sin(2t)]$. The plant is assumed to be *uncertain*, only norm bounds for $\theta^*$ and $\phi$ are known. Assumption (I) is trivially satisfied, since $\phi$ is triangular. In (II): $k_x = 1$, $\varphi = \Psi_\varphi = y^2$ and $k_\varphi = 0$.

In (4), $\Lambda = \mathrm{diag}\{-1, -2\}$ and $g^{\mathrm{T}} = [1\ 1]$. The modulation function $f(t)$ is implemented using (11), (12) and (22), with $\bar{\theta}^{\mathrm{T}} = [2\ 2\ 30\ 15\ 10\ 5]$ and $\delta = 0.1$. To compute all the FOAF's parameters involved in (11) and (12), i.e. $c_1 = 1, c_2 = 2$, $\lambda_x = 0.8$, $c_\phi = 1$ and $\gamma_\phi = 0.8$, one can use a simple technique based on Lyapunov quadratic forms, or a less conservative one based on optimization methods [21]. Other design parameters are included in Table I.



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The nonlinearity $\phi$ reflected to the input, as $d_\phi$ in (8), is given by

$$d_\phi = 2(x_2 + y^2)^2 + 12y(x_2 + y^2) + 2y(x_3 + 2yx_2 + 2y^3) + 12y^2 + x_2 \sin(2t) + \bar{W}_\phi * \phi \quad (37)$$

Since $y$ is measured, the class-$\mathscr{K}$ $\Psi_\phi$ (12) is replaced by the less conservative function:

$$\bar{\Psi}_\phi(\hat{x}, |y|) = 2(\hat{x} + y^2)^2 + 12|y|(\hat{x} + y^2) + 2|y|(\hat{x} + 2|y|\hat{x} + 2|y|^3) + 12y^2 + \hat{x} \quad (38)$$

The plant initial conditions are $y(0) = 10$, $\dot{y}(0) = 10$ and $\ddot{y}(0) = 10$ and a wrong control direction estimate is assumed at $t = 0$. Figure 4(a) shows that after the peak extinction time $\bar{t}_e$ just one switching in the control sign is needed (second jump of $\varphi_m$ when it meets $|\hat{e}_g|$). After that, the control direction is correctly identified and $\hat{e}_g$ vanishes in finite time. Figure 4(b) shows the changes between the linear lead filter ($\alpha = 1$) and RED ($\alpha = 0$). It is clear that RED is ultimately chosen by the switching strategy of the hybrid lead filter. This is translated by the convergence of the plant output to the model reference output signal in Figure 4(c). Figure 4(d) and (e) points out the remarkable transient response of the proposed scheme under time-varying control direction. In Figure 4(d), the transient response of $e$ stays almost unaltered after changes in the control direction at $t = 0$, 2, 4, 6 and 8 s which can be observed from Figure 4(e).

## 12. CONCLUSIONS

An output-feedback model-reference sliding mode controller was developed for nonlinear uncertain systems with unknown HFG sign and arbitrary relative degree, generalizing the controller introduced in [10]. The resulting controller leads to global or semi-global asymptotic stability with respect to some compact set and ultimate exponential convergence of the tracking error to zero. Simulations illustrate the performance of the proposed scheme under polynomial output-dependent unmatched disturbance. Experimental results can be found in [15]. Further developments of the proposed scheme could include the analysis of the case of non-constant (time or state dependent) HFG, verified to be possible by simulation and experiments. The assumptions made in this paper about the plant are satisfied by the well-known class of triangular systems. However, it would be desirable to characterize more general systems that could be included in the proposed approach.

## APPENDIX A

### *A.1. Auxiliary lemma*

*Lemma 1*

Consider an arbitrary initial time $t_i \in [0, t_M)$ and the I/O relationship

$$\varepsilon(t) = \frac{\bar{k}}{(s + \bar{\alpha})}[u + d(t)] + \pi(t) + \beta(t), \quad \bar{\alpha} > 0 \ \ \forall t \in [t_i, t_M) \quad (A1)$$

where $\text{sgn}(\bar{k})$ is known, $u = -[\text{sgn}(\bar{k})] f(t) \,\text{sgn}(\varepsilon)$, $f(t)$ and $d(t)$ are locally integrable in the sense of Lebesgue (LI), $\beta(t)$, $\pi(t)$, $\gamma(t)$ are absolutely continuous $\forall t \in [0, t_M)$,

$$\gamma(t) \geqslant |\beta(t)| + |\pi(t)| \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \geqslant -\bar{\alpha}\gamma(t) \quad \forall t \in [t_i, t_M) \quad (A2)$$



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If the modulation function $f(t)$ satisfies $f(t)\geqslant|d(t)|, \forall t\in[t_i, t_M)$, then the signals $\varepsilon(t)$ and $\bar{e}(t):=\varepsilon(t)-\beta(t)-\pi(t)$ are bounded by

$$|\bar{e}(t)|, |\varepsilon(t)|\leqslant|\varepsilon(t_i)-\beta(t_i)|\mathrm{e}^{-\bar{\alpha}(t-t_i)}+2\gamma(t) \quad \forall t\in[t_i, t_M) \tag{A3}$$

*Proof*
See the proof of [22, Lemma 2]. □

*A.2 Proof of Theorem 1*

In what follows, $k_i, \bar{k}_i>0$ are constants not depending on the initial conditions and $\Psi_i(\cdot)\in\mathscr{K}$.

*ISpS-like property from $z$ to $\bar{\beta}$*: Let $x_m^{\mathrm{T}}:=[y_m\ \dot{y}_m\ \ldots\ y_m^{(n^*-1)}]$ and $x_e:=\xi-x_m$, with $\xi$ in (I). From (9), it can be shown that $\mathrm{e}^{(i)}=h_{\mathrm{c}}^{\mathrm{T}}A^iX_e, i=0,\ldots,n^*-1$, hence $|x_e|\leqslant k_0|X_e|$. Therefore, since $x_m$ is uniformly bounded, then $\xi$ can be norm bounded affinely in $|X_e|$. By (I), there exists a normalizing global diffeomorphism and the $\eta$-dynamics is ISS w.r.t. $\xi$. Thus, one can conclude that $|x|\leqslant\Psi_1(\|\xi_t\|)+k_1$, and consequently, $|x|\leqslant\Psi_2(\|(X_e)_t\|)+k_2$. Now, from (II), (7), and the bound of $d_\phi$ given in (12) one has $|X_m|\leqslant\Psi_3(\|(X_e)_t\|)+k_3$. Then, reminding that $\omega=\Omega_1X+\Omega_2r$, and since $X=X_e+X_m$, one can verify that $|\omega|, |f|\leqslant\Psi_4(\|(X_e)_t\|)+k_4$, with $f(t)$ of (22). Moreover, from the small norm property of $W_\beta(s,\tau)$ in (26), one can conclude that $\|\bar{\beta}_t\|\leqslant\tau\Psi_5(\|z_t\|)+\mathscr{O}(\tau)$, since $|X_e|\leqslant|z|$. By continuity, given any $R>0$, if $|z(0)|\leqslant R/2$ $\exists t^*\in[0,t_M)$ such that $|z(t)|<R$, $\forall t\in[0,t^*)$. Hence, $\forall t\in[0,t^*)$, $\Psi_i(a)\leqslant k^Ra$, with $k^R>0$ constant and possibly depending on $R$. Therefore,

$$\|\bar{\beta}_t\|\leqslant\tau k^R\|z_t\|+\mathscr{O}(\tau) \quad \forall t\in[0,t^*) \tag{A4}$$

*Upper bound for $z$ during the peaking* ($\forall t\in[0,t_1]$): Reminding that the peaking extinction time satisfies $t_e\leqslant\bar{t}_e(\tau)$, where $\bar{t}_e(\tau)\in\mathscr{K}$, then $\bar{t}_e(\tau)<t^*$, for $\tau$ sufficiently small. For $t\in[0,t_1]$, where $t_1=\bar{t}_e$ is the first switching time, $u-\bar{u}$ is affinely bounded by $\|(X_e)_t\|$ and the solution of (9) can diverge at most exponentially according to $|X_e(t)|\leqslant\mathrm{e}^{k_Lt}|X_e(0)|+k_5$ for some positive constants $k_L$ and $k_5$. Therefore, also taking into account (20), one has

$$|z(t)|\leqslant k_1^R|z(0)|+\mathscr{O}(\tau) \quad \forall t\in[0,t_1] \tag{A5}$$

*Upper bound for $z$* $\forall t\in[0,t_M)$: From (29), (32), (33) and (35), one has $\|(\hat{e}_g)_{t,t_1}\|\leqslant|\hat{e}_g(t_p)|+a(k)+3\|\bar{\beta}_t\|$, $\forall t\in[t_1,t_M)$, where $k\geqslant1$ is such that $t\in[t_k,t_{k+1}]$, and $p=\mathrm{argmax}_{i\in\{1,2,\ldots,k\}}|\hat{e}_g(t_i)|$. With $z$ from (36), (28) implies $\|(e_F^0)_{t,t_1}\|\leqslant k_5|z(0)|$. Thus, since $|\beta|\leqslant\bar{\beta}$ and $\bar{e}=\hat{e}_g-\beta-e_F^0$ (see (13) and (25)), one has

$$\|(\bar{e})_{t,t_1}\|\leqslant|\bar{e}(t_p)|+a(k)+2k_5|z(0)|+5\|\bar{\beta}_t\| \quad \forall t\in[t_1,t_M) \tag{A6}$$

Since $ML(s)=k_m/(s+a_m)$, from (13) and (9) one gets $\dot{X}_e=A_{\mathrm{c}}X_e+(b_{\mathrm{c}}/k_m)(\dot{\bar{e}}+a_m\bar{e})$. Further, using the simple transformation $X_e:=\bar{X}_e+(b_{\mathrm{c}}/k_m)\bar{e}$, one gets $\dot{\bar{X}}_e=A_{\mathrm{c}}\bar{X}_e+(A_{\mathrm{c}}b_{\mathrm{c}}+a_mb_{\mathrm{c}})\bar{e}$ which clearly implies an ISS relationship from $\bar{e}$ to either $X_e$ or $\bar{X}_e$. Moreover, since $A_f$ in (20) is Hurwitz, this system is ISS w.r.t. $e=h_{\mathrm{c}}^{\mathrm{T}}X_e$ and also to $\bar{e}$. Thus, $z$ (36) satisfies an upper bound similar to (A6), valid $\forall t\in[t_1,t_M)$. Now, taking into account (A5) and noting that $|z(t_p)|\leqslant\max_{i=1,\ldots,k}\{|z(t_i)|\}$, one can conclude that

$$\|(z)_t\|\leqslant k_8\max_{i=1,\ldots,k}\{|z(t_i)|\}+k_9a(k)+k_7|z(0)|+k_{10}\|\bar{\beta}_t\|+\mathscr{O}(\tau) \quad \forall t\in[0,t_M) \tag{A7}$$



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*The control direction switching stops*: Suppose that $u$ (21) switches between $u^+$ and $u^-$ without stopping, $\forall t \in [0, t^*)$. Then, $a(k)$ in (33) increases unboundedly as $k \to \infty$. Thus, there is a finite value $k = \kappa$ such that $a(\kappa) \geqslant 2R_a e^{\bar{\lambda}_a \bar{t}_e}$ (see (28)) and $\mathrm{sgn}(k_p)$ is correctly estimated. In this case, $\varphi_m(t) > \zeta(t)$, $\forall t \in [t_\kappa, t_{\kappa+1})$, with $\zeta$ in (31). Moreover, $\zeta$ is a valid upper bound for $|\hat{e}_g|$. Hence, no switching will occur after that until $t = t^*$ which leads to a contradiction. Therefore, $\varphi_m$ (33) has to stop switching after some finite $k = N$ for $t \in [0, t^*)$.

*Stability w.r.t. a compact set*: It is not difficult to conclude that $N$ can be related to $|z(0)|$, since $R_a \leqslant \bar{k}_1 |z(0)|$ by definition. Indeed, one can write $N \leqslant \Psi_6(|z(0)|) + \bar{k}_2$. Thus, one has $a(N) \leqslant \Psi_7(|z(0)|) + \bar{k}_3$ and, from (A7), $\|(z)_t\| \leqslant k_8 \max_{i=1,\ldots,k}\{|z(t_i)|\} + \Psi_8(|z(0)|) + k_{10}\|\bar{\beta}_t\| + \mathcal{O}(\tau) + \bar{k}_4$, $\forall t \in [0, t^*)$. Thus, from (A4), which contains a small gain loop, one has

$$|z(t_{k+1})|,\ \|(z)_t\| \leqslant k_8 \max_{i=1,\ldots,k}\{|z(t_i)|\} + \Psi_9(|z(0)|) + \mathcal{O}(\tau) + \bar{k}_5 \quad \forall t \in [0, t^*) \tag{A8}$$

provided $\tau < 1/(k^R k_{10})$. In addition, from the recursive inequality in (A8) and from (A5) one can obtain $\|(z)_t\| \leqslant \Psi_{10}(|z(0)|) + c_z$, $\forall t \in [0, t^*)$, where $c_z$ is a positive constant. Thus, given $R > c_z$, for $|z(0)| < R_0$, with $R_0 \leqslant \Psi_{10}^{-1}(R - c_z)$, then $|z(t)|$ is bounded away from $R$ as $t \to t^*$. This implies that $z(t)$ is uniformly bounded and cannot escape in finite time, i.e. $t_M = +\infty$. Hence, stability with respect to the ball of radius $c_z$ is guaranteed for $z(0)$ in the $R_0$-ball. Since $R$ and thus $R_0$ can be chosen arbitrarily large as $\tau \to 0$, semi-global stability is concluded. Moreover, if $\phi$ is globally Lipschitz and/or $n^* = 1$ then the stability properties become global.

*Exponential convergence to a small residual set*: Independently of whether the control direction is correctly found or not, at $k = N$, take $t = t_N$ as a new initial instant of time. Now, $z$ can be bounded using (35), the ISS norm bound of $z(t)$ in terms of $\bar{e} = \hat{e}_g - \beta - e_F^0$ and (A4) to get the inequalities ($\forall t \geqslant t_N$):

$$|z(t)| \leqslant \bar{k}_6[|z(t_N)| + a(N)]e^{-\lambda_1 t} + \bar{k}_7 \|\bar{\beta}_{t,t_N}\| \quad \text{and} \quad |\bar{\beta}(t)| \leqslant \tau k_2^R \|z_{t,t_N}\| + \mathcal{O}(\tau) \tag{A9}$$

where $0 < \lambda_1 \leqslant \min(a_m, \lambda_c)$ and $k_2^R$ is a positive constant possibly depending on $R$. Then, applying the *small-gain theorem* [16], one has that $|z(t)| \to \mathcal{O}(\tau)$ exponentially, as $t \to +\infty$.

*Switching stops at a correct sign*: Since the error state $z$ enters the $\mathcal{O}(\tau)$ residual set, the exact differentiator will eventually take over providing the exact estimate of the ideal sliding variable $\bar{e}$, i.e. $\hat{e}_g = \bar{e}$. Suppose that we end up with an incorrect control direction estimate. Then, the equation for $\bar{e}$ can be written as $\dot{\bar{e}} = a_m \bar{e} + |k_p|[f(t)\,\mathrm{sgn}(\bar{e}) - u^\dagger] + \pi$, where $a_m > 0$ and $u^\dagger = \omega^{\mathrm{T}}\theta^\dagger - d_\phi$. In this case, due to (22) ($f > u^\dagger + \delta$), there exists $t_d < +\infty$ such that $|\pi| < \delta$ and, consequently, $\bar{e}\dot{\bar{e}} > 0$, $\forall t \geqslant t_d$. Hence $\bar{e}$ would diverge as $t \to \infty$ for all initial conditions, i.e. $\bar{e}$ would not remain in the residual set. Therefore, sgn $(k_p)$ must be correctly estimated at $k = N$.

*Exact tracking*: The sliding variable $\bar{e}$ and the control direction are exactly estimated. Then, from [22, Lemma 1], the ideal sliding mode $\bar{e} \equiv 0$ is achieved in finite time. Further, the full error state $z$, as well as the tracking error $e$, converge exponentially to zero. $\square$

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