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Sliding Mode Control of Uncertain Nonlinear Systems with Arbitrary Relative Degree and Unknown Control Direction.

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***Abstract*— This paper considers the model reference tracking control for a class of uncertain nonlinear systems, based on sliding mode and output-feedback. No particular growth condition is imposed on the nonlinearity. Moreover, the design does not assume the *prior* knowledge of the control direction. For plants of arbitrary relative degree, global or semi-global asymptotic stability with respect to a compact set is guaranteed. Ultimate finite-time or exponential convergence of the tracking error to zero is achieved by using a hybrid lead filter based on 2-sliding mode exact differentiators. A monitoring function is used to determine the unknown control direction.**

Keywords: uncertain nonlinear systems, output feedback, control direction, exact tracking, exact differentiators, 2-sliding mode.

I. Introduction

The problem of controlling uncertain plants with unknown control direction, i.e., the sign of the high frequency gain, has attracted the attention of the adaptive control community since the early 1980's [3]. A solution to the problem appeared in [4], where the so called Nussbaum-type gains were introduced to design stable adaptive control systems under this relaxed assumption. This concept became a standard design tool in adaptive control theory as in [5], and more recently in [6], [7], [8].Although in theory, this approach may lead to a solution of the problem, it is well known that the resulting transient behavior can be unacceptable [5][9].

For sliding mode control (SMC) with unknown control direction, fewer results are available In [10], a state feedback sliding mode controller was proposed for a class of uncertain nonlinear systems without need of explicitly identifying the sign of the control direction. In [11], a hybrid scheme was proposed for uncertain first order nonlinear systems with hard nonlinearities, avoiding the large peaking transient resulting from the Nussbaum gain approach. An output feedback SMC scheme for tracking of uncertain linear plants was introduced in [1] utilizing a switching algorithm based on a monitoring function for the output tracking error. Similar controller was extended to nonlinear systems in [2] where the Model Reference Robust Control approach was adopted. However, [1] and [2] approach only relative degree one plants.

In this paper, we further extend the later results to the case of nonlinear plants with arbitrary relative degree, using only output feedback. It is desired to obtain global or semi-global stability and exact tracking without knowledge of the control direction. To this end, we use a hybrid scheme [16] which combines conventional linear lead filter with exact 2-sliding differentiators [18]. The lack of knowledge of the control direction is circumvented by a switching mechanism that adjusts the control sign, being driven by a monitoring function for an appropriate auxiliary error.

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II. Problem Formulation

Consider an uncertain SISO LTI plant

$$y = G_p(s)[u + d_e(y,t)]\,, \tag{1}$$

where u is the control input, y is the output, $d_e(y,t)$ is a matched input disturbance and $G_p(s) = k_p(N_p(s)/D_p(s))$, with $N_p(s)$ and $D_p(s)$ being monic polynomials of degree m and n, respectively. The following assumptions are made: (A1) $G_p(s)$ is minimum phase, strictly proper and its parameters are unknown but belong to a known compact set. (A2) The degree n of $D_p(s)$ is a known constant. (A3) $G_p(s)$ has known relative degree $n^* := n - m$. The above Assumptions (A1)–(A3) are usual in adaptive control [15]. Consider the following additional assumptions:

(A4) The sign of the high frequency gain $k_p \neq 0$ is unknown.

(A5) The disturbance $d_e(y,t)$ is locally Lipschitz in y, $\forall y$, and piecewise continuous in t, $\forall t$.

(A6) The nonlinear disturbance $d_e(y,t)$ satisfies

$$|d_e(y,t)| \leq \bar{d}_e(y,t)\,, \quad \forall (y,t)\,,$$

where $\bar{d}_e : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a known function piecewise continuous in t and continuous in y, satisfying $\bar{d}_e(y,t) \leq \Psi(|y|) + k_\Psi$, where $\Psi \in \mathcal{K}_\infty$ and $k_\Psi > 0$ is a constant.

(A4) represents the relaxation of the classical assumption on the *prior* knowledge concerning sgn(k_p); (A5) allows us to develop a control law u that guarantees local existence and uniqueness (in positive time) of the solution of (1).

Remark 1: In (A6), no particular growth condition is imposed on d_e, e.g., $d_e(y,t) = y^2$. Since finite-time escape is not precluded, *a priori*, $[0, t_M)$ is defined as the maximum time interval of definition of a given solution, where t_M may be finite or infinite.

Reference Model: the reference model is given by

$$y_m = M(s)r = (k_m/D_m(s))r\,,\ k_m > 0\,, \tag{2}$$

where the reference signal $r(t)$ is assumed piecewise continuous and uniformly bounded, D_m is a monic polynomial of degree n^*.

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Control Objective: the control objective is to achieve global or semi-global stability and convergence of the error state with respect to the origin of the error space. In particular, the tracking error

$$e_0(t) = y(t) - y_m(t) \tag{3}$$

should asymptotically tend to zero, i.e., exact tracking is required.

Notation: In this paper we adopt the following notation. The 2-norm (Euclidean) of a vector x and the corresponding induced of a matrix A are denoted by $|x|$, or $|A|$, respectively. The $\mathcal{L}_{\infty e}$ norm of the signal $x(t) \in \mathbb{R}^n$ is defined as $\|x_{t,\bar{t}_0}\| := \sup_{\bar{t}_0 \leq \tau \leq t} |x(\tau)|$. For $\bar{t}_0 = 0$ the notation is adopted $\|x_t\| := \sup_{0 \leq \tau \leq t} |x(\tau)|$. The symbol "$s$" represents either the Laplace variable or the differential operator "d/dt", according to the context. As in [15], [14] the output y of a linear time invariant system with transfer function $H(s)$ and input u is given by $y = H(s)u$. Convolution operations $h(t) * u(t)$, with $h(t)$ being the impulse response from $H(s)$, will be eventually written, for simplicity, as $H(s)*u$. The *stability margin* λ_0 of a transfer function $G(s)$ is *defined* as $\lambda_0 := \min_i\{-Re(\lambda_i)\}$ where $\{\lambda_i\}$ are the poles of $G(s)$.

A. Output Error Equation

Considering the usual model reference adaptive control (MRAC) approach [15], the output error e_0 satisfy (see [12] for details)

$$e_0 \quad = \quad k^* M(s)[u - u^*] \,, \tag{4}$$

where $k^* = k_p/k_m$,

$$u^* := \theta^{*T}\omega - W_d(s) * d_e \,, \tag{5}$$

is the model matching control in the presence of d_e. The *regressor vector* ω is composed by the states of the input/output filters [15], by the plant output y and by the reference signal r. The *ideal parameter vector* θ^* is unknown but is elementwise bounded by a known constant vector $\bar{\theta}^T$ ($\bar{\theta}_i > |\theta_i^*|$) [12]. The transfer function $W_d(s) = [k^* M(s)]^{-1} \bar{W}_d$ is proper and stable where $\bar{W}_d(s)$ is the closed-loop transfer function from the input disturbance d_e to e_0 (see [14] for details).

The signal u^* will be regarded as a matched input disturbance, thus an upper bound will be required. Since W_d is a proper and BIBO stable transfer function and d_e satisfies Assumption (A6), then applying [19, Lemma 2] to the convolution $W_d(s) * d_e(y,t)$, one can find positive constants c_d, γ_d such that $|W_d(s) * d_e(y,t)| \leq \hat{d}_e(t)$, where $\hat{d}_e$ is defined by

$$\hat{d}_e(t) := \bar{d}_e(y,t) + c_d e^{-\gamma_d t} * \bar{d}_e(y,t) \,. \tag{6}$$

Thus, from (5), u^* satisfies

$$|u^*(t)| \leq \bar{\theta}^T |\omega(t)| + \hat{d}_e(t) \,,\ \forall t \in [0, t_M) \,. \tag{7}$$

III. The Case of Plants with Relative Degree One

Consider the case of relative degree one, unknown $\mathrm{sgn}(k_p)$, and nonlinear disturbances. This section will generalize the results of [1] developed for linear plants.

The control law is defined by

$$u = \begin{cases} u^+ = -f(t)\ \mathrm{sgn}(e_0) & ,\quad t \in T^+ \,, \\ u^- = f(t)\ \mathrm{sgn}(e_0) & ,\quad t \in T^- \,, \end{cases} \tag{8}$$

where an appropriate monitoring function [1] of the tracking error e_0 is used to decide when u would be switched from u^+ to u^- and vice versa, allowing the detection any wrong estimate of $\mathrm{sgn}(k_p)$. The sets T^+ and T^- satisfy $T^+ \cup T^- = [0, t_M)$ and $T^+ \cap T^- = 0$, and as will be shown in the following analysis, both T^+ and T^- have the form $[t_k, t_{k+1}) \cup \cdots \cup [t_j, t_{j+1})$. Here, t_k or t_j denotes the switching time for u and will be defined later. We refer to such switchings as *control sign switchings*.

According to (4), the *modulation function* $f(t)$ should be a norm bound of u^*. From (7), one possible choice is

$$f(t) = \bar{\theta}^T |\omega(t)| + \hat{d}_e(t) + \delta \,, \tag{9}$$

where δ is an arbitrary nonnegative constant. Consider for simplicity $M(s) = k_m/(s + a_m)$ ($a_m, k_m > 0$). Then for $\mathrm{sgn}(k_p)$ known, one chooses the control u^+ or u^-, according to $k_p > 0$ or $k_p < 0$, respectively. Now, e_0 satisfies

$$\dot{e}_0(t) \quad = \quad -a_m e_0(t) + k_p[u(t) - u^*(t)] + \pi(t) \,, \tag{10}$$

where $\pi(t)$ denotes a transient term due to initial conditions of the observable but not controllable subsystem of the nonminimal realization (A_c, b_c, h_c^T) of $M(s)$ in (4), used in MRAC theory [15]. Now, noting that $\mathrm{sgn}(u - u^*) = -\mathrm{sgn}(e_0)$, if the correct control direction is used and $f(t) > |u^*|$, then by using the *Comparison Theorem* [13], $|e_0|$ is bounded by the solution of the following differential equation

$$\dot{\xi}(t) = -a_m \xi(t) + \pi(t),\ \forall t \in [\bar{t}_0, t_M),\ \xi(\bar{t}_0) = e_0(\bar{t}_0) \,, \tag{11}$$

i.e., $\forall t \geq [\bar{t}_0, t_M)$, one has

$$|e_0(t)| \leq |\xi(t)| \leq e^{-a_m(t - \bar{t}_0)} |e_0(\bar{t}_0)| + c_0 e^{-\delta t} \,, \tag{12}$$

where $\bar{t}_0$ denotes some initial time.

A. Monitoring Function ($n^* = 1$)

Based on (12), consider the auxiliary function φ_k defined as follows:

$$\varphi_k(t) = e^{-a_m(t - t_k)} |e_0(t_k)| + (k+1) e^{-\frac{t}{k+1}} \,, \tag{13}$$

$$t \in [t_k, t_M),\ t_0 := 0,\ (k = 0, 1, \ldots) \,.$$

The *monitoring function* φ_m can be defined as

$$\varphi_m(t) \quad := \quad \varphi_k(t)\,, \forall t \in [t_k, t_{k+1}) (\subset [0, t_M)) \,. \tag{14}$$

The motivation behind the introduction of φ_m is that π is not available for measurement. Reminding that the inequality (12) holds if the $\mathrm{sgn}(k_p)$ is correctly estimated, it seems natural to use ξ as a benchmark to decide whether a switching of u is needed. However, since π is not available, one has to use φ_m to replace ξ and invoke the switching of φ_m. Note that from (14), one always has $|e_0(t_k)| < \varphi_k(t_k)$ at $t = t_k$. Hence, the switching time t_k for u from u^- to u^+ (or u^+ to u^-) is well-defined (for $k \geq 0$):

$$t_{k+1} = \begin{cases} \min\{t > t_k : |e_0(t)| = \varphi_k(t)\}, & \text{if it exists}, \\ t_M, & \text{otherwise}. \end{cases} \quad (15)$$

B. Main Result for $n^* = 1$

Theorem 1: Assume that (A1)–(A6) hold. Consider the system defined by (1), (2) and (8) and the modulation function given in (9). Then, the control sign switchings, driven by the monitoring function (14), will stop after a finite number of switchings and both the tracking error e_0 and the complete state X_e will converge to zero at least exponentially.

Proof: We only sketch the proof, which is divided in three parts. First it is proved that the switching stops after a finite number of switchings (avoiding finite-time escape), since for some finite k^* the term $(k^* + 1)e^{-t/(k^*+1)}$ of (13) will allow $\varphi_k(t)$ to be an upper bound valid for ξ, in (12), consequently no switching will occur after that. Second, if the control direction is correctly estimated or not, since φ_k converges to zero exponentially $e_0(t)$ will also converge to zero, at least exponentially, avoiding finite-time escape. Finally, the convergence of the complete error state X_e can be shown by using the regular form for the state space realization of (4). ■

Corollary 1: In Theorem 1, the control sign switching stops at a correct sign corresponding to the unknown sign of the control direction of the plant, i.e., for $t > t_{k^*}$, $u = u^+$, if $k_p > 0$ and $u = u^-$, otherwise.

Proof: The proof is based on a reverse dynamics argument. We know that if the sign is correct all trajectories of the system converge to the origin of the error state space (Lemma 1 in [14]).

Reverse Dynamics Argument: Assume that the final control sign is incorrect. Then, if we reverse the time, i.e., $t \to -t$, the resulting equations have the same stability properties as those obtained with the right control sign and thus all trajectories from any initial condition would converge to the origin, i.e., the origin would be a global sink in reverse time. Thus, in forward time, all trajectories not at the origin would diverge unboundedly. This is a contradiction, since by Theorem 1 the state converges to the origin. Thus, the ultimate control sign must be correct. ■

IV. The Case of Plants with Arbitrary Relative Degree

The main idea for generalizing the previous case consists in reducing the problem to the $n^* = 1$ case by the introduction of the operator

$$L(s) = s^N + a_{N-1}s^{N-1} + \ldots + a_0\,, \quad N := n^* - 1\,, \quad (16)$$

such that $G_pL(s)$ be of relative degree one (or, equivalently, almost strictly positive real – ASPR) and $ML(s)$ be SPR (or ASPR). However, $L(s)$ is non-causal and what can be actually implemented is an approximate realization of this operator. One approximation is $\mathcal{L}$ given by the linear lead filter

$$\mathcal{L}(s) = L(s)/F(\tau s)\,, \quad F(\tau s) = (\tau s + 1)^N \text{ and } \tau > 0\,, \quad (17)$$

As will be shown, this approximation leads to global/semi-global stability with respect a residual set of order $\mathcal{O}(\tau)$. However, it is well known that such filters usually lead to control chattering and nonzero residual tracking error due to the phase lag introduced the time constant (τ). Alternatively, $L(s)$ can be implemented by using the Levant's robust exact differentiators (RED) [18] which potentially allows the exact estimate of the e_0 derivatives. The problem is that such differentiators are valid only locally and may lead to unstable behavior with larger initial conditions [16].

In the proposed control strategy, see Figure 1, $L(s)$ is replaced by a *hybrid lead filter*, named Global Robust Exact Differentiator (GRED) [16]. In Fig. 1, α represents a switching law. It is then possible to obtain a exact compensation of the relative degree while assuring global or semi-global stability properties of the closed loop system. The control sign is adjusted according to the monitoring function φ_m, as indicated in Fig. 1.

The control u is defined as in (8), replacing e_0 by $\tilde{\varepsilon}_0 := \alpha\bar{\varepsilon}_0 + (1-\alpha)\varepsilon_0$ (see Fig. 1), i.e.,

$$u = \begin{cases} u^+ = -f(t)\ \text{sgn}(\tilde{\varepsilon}_0) & , \quad t \in T^+\,, \\ u^- = f(t)\ \text{sgn}(\tilde{\varepsilon}_0) & , \quad t \in T^-\,, \end{cases} \quad (18)$$

The strategy for switching the control direction, according to a new monitoring function φ_m, will be redefined later on.

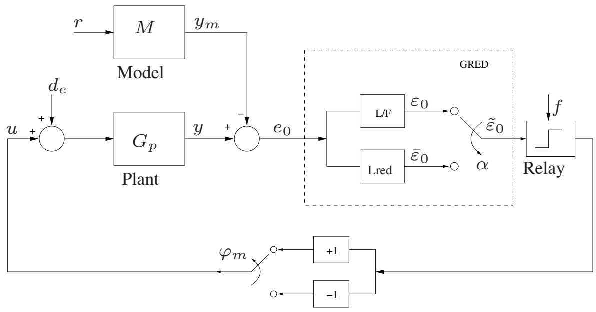

Fig. 1. VS-MRAC using a hybrid lead filter (GRED) for relative degree compensation and a switching scheme to adjust the control sign. LF and L_{red} stand, respectively, for the linear and nonlinear (RED based) lead filters.

Equivalent Structure for the Hybrid Lead Filter: According to [16, Lemma 2], choosing an appropriate $\alpha(\cdot)$, the GRED is *equivalent* to a linear lead filter with an uniformly bounded output disturbance β_α of order τ, modulo exponentially decaying terms. However, in order to simplify the analysis, we will meanwhile ignore β_α. This corresponds to $\tilde{\varepsilon}_0 = \varepsilon_0$. The analysis will be easily completed in Subsection IV-C.1, considering the neglected output disturbances.

A. Auxiliary Errors for Analysis and Design

As explained above, assume that only the linear lead filter is active, i.e., $\tilde{\varepsilon}_0 = \varepsilon_0$. Then, from Figure 1, one has

$$\varepsilon_0 \quad = \quad \frac{L(s)}{F(\tau s)} e_0 \,, \tag{19}$$

which can be rewritten as

$$\varepsilon_0 \quad = \quad k^* ML \left[u - u^* \right] + \beta_{\mathcal{U}} + e_F^0 \,, \ \forall t \in [0, t_M) \,, \tag{20}$$

where

$$\beta_{\mathcal{U}} := k^* ML(s) \left[1 - F(\tau s) \right] F^{-1}(\tau s) * (u - u^*) \text{ and} \tag{21}$$

$$|e_F^0| \leq R_1 e^{-\lambda_c t} + \frac{R_2}{\tau^N} e^{-\frac{t}{\tau}} \leq R_a e^{-\lambda_a (t - t_e(\tau))} \,. \tag{22}$$

The positive constants R_1, R_2, R_a and λ_c are *independent* of $\tau > 0$; λ_c is lower than the stability margin of A_c and $0 < \lambda_a < \min(\lambda_c, 1/\bar{\tau})$, with $\bar{\tau} > \tau$.

The first inequality in (22) holds $\forall t \geq 0$, while the last one holds only $\forall t \geq t_e$ where t_e is the *peak extinction time*, i.e., the smallest time value at which the inequality $\frac{R_2}{\tau^N} e^{-\frac{t}{\tau}} \leq R_2, \forall t \geq t_e(\tau), \forall R_2$ is satisfied for a fixed value of the parameter $\tau \in (0, 1]$.

The constants R_1 and R_2 are linear combination of the initial conditions $X_e(0)$ and $x_f(0)$, where x_f is the state vector of the realization $(\frac{A_f}{\tau}, \frac{B_f}{\tau}, \frac{C_f}{\tau^N}, \frac{1}{\tau^N})$ with $(A_f, B_f, C_f, 1)$ being the canonical controllable realization of L/F in (19). By using this realization, peaking appears only in the output ε_0 while the state x_f is peaking free.

An Upper Bound for t_e (peak extinction time): It can be easily concluded that $t_e(\tau)$ is uniformly bounded by a class-$\mathcal{K}$ function of τ. Moreover, there exist $\bar{t}_e(\tau) \in \mathcal{K}$ such that

$$t_e(\tau) \leq \bar{t}_e(\tau) \,, \tag{23}$$

which can be obtained from the known upper bounds of the plant parameters.

Considering the error system (4), (19), the following state vector z is used

$$z^T := [X_e^T, x_f] \,, \ z \in \mathbb{R}^{3n-2+N} \,. \tag{24}$$

The following inequality is a consequence of the continuity of the Filippov solutions and the particular state realization associated with x_f:

$$|z(t)| \quad \leq \quad k_{z0} |z(0)| + \mathcal{V}(\tau) \,, \tag{25}$$

$\forall t \in [0, t_e(\tau)] \subset [0, t_M)$, $\forall \tau \in (0, \tau_1]$; $0 < \tau_1 \leq 1$; $\mathcal{V} \in \mathcal{K}$ and $k_{z0} > 0$ is a constant.

B. Monitoring Function ($n^* > 1$)

The following lemma provides an upper bound for $|\varepsilon_0|$, valid if $\text{sgn}(k_p)$ is known and $t \in [\bar{t}_e, t_M)$, from which the new monitoring function will be defined.

Lemma 1: Consider the I/O relationship

$$\varepsilon(t) = \bar{M}(s)[u + d(t)] + \pi(t) + \beta(t) \,, \tag{26}$$

and any arbitrary initial time $\bar{t}_0 \geq 0$, where $\bar{M}(s) = \bar{k}/(s + \bar{\alpha})$ ($\bar{k}, \bar{\alpha} > 0$), $d(t)$ is LI, $\beta(t)$ and $\pi(t)$ are absolutely continuous, $\forall t \in [\bar{t}_0, t_M)$. Assume that $|\pi(t)| \leq Re^{-\lambda(t - \bar{t}_0)}, \forall t \in [\bar{t}_0, t_M)$, where R, λ are positive constants. If $u = -f(t) \, \text{sgn}(\varepsilon)$, where the modulation function $f(t)$ is LI and satisfies $f(t) \geq |d(t)|, \forall t \in [\bar{t}_0, t_M)$, then the signal $\bar{e}(t) := \varepsilon(t) - \beta(t) - \pi(t)$ is bounded by (for any arbitrary t_i such that $\bar{t}_0 \leq t_i < t_M$ and $\bar{\alpha}_\lambda := \min(\bar{\alpha}, \lambda)$)

$$|\bar{e}(t)| \leq |\varepsilon(t_i) - \beta(t_i)| e^{-\bar{\alpha}(t - t_i)} + Re^{-\bar{\alpha}_\lambda (t - \bar{t}_0)} + \|\beta_{t, \bar{t}_0}\| \,. \tag{27}$$

Proof: The proof is similar to the proof of [14, Lemma 2]. ■

Reminding that $\varepsilon_0 = \beta_{\mathcal{U}} + \bar{e}_0 + e_F^0$ then $|\varepsilon_0| \leq |\beta_{\mathcal{U}}| + |\bar{e}_0| + |e_F^0|$. Now, applying Lemma 1 to (20), considering $\bar{t}_0 := \bar{t}_e$ and $ML(s) = k_m/(s + a_m)$ (for simplicity), and from (22) one has $\forall t, t_k$ such that ($t_M > t \geq t_k \geq \bar{t}_e$),

$$\begin{aligned} |\varepsilon_0(t)| \quad &\leq \quad (|\varepsilon_0(t_k)| + |\beta_{\mathcal{U}}(t_k)|) e^{-a_m (t - t_k)} + \\ &+ \quad (2R_a e^{\bar{\lambda}_a \bar{t}_e}) e^{-\bar{\lambda}_a t} + 2\|(\beta_{\mathcal{U}})_{t, \bar{t}_e}\| \,, \end{aligned} \tag{28}$$

where $\bar{\lambda}_a = \min\{a_m, \lambda_a\}$. Note that, according to Lemma 1, (28) is valid for the modulation function $f(t)$ given in (9).

Consider the available signal

$$\bar{\beta}_{\mathcal{U}} = 2\bar{k}^* \tau W_\beta(s) * f(t) \,, \tag{29}$$

where $\tau W_\beta(s)$ is a first order approximation filter (FOAF, [19]) for the transfer function $ML(s) \left[1 - F(\tau s) \right] F^{-1}(\tau s)$. Note that, from (21), (18) and (9), one has $\beta_{\mathcal{U}}(t) \leq \bar{\beta}_{\mathcal{U}}(t)$ ($\forall t \in [0, t_M)$). Let

$$\begin{aligned} \varphi_k(t) \quad &:= \quad (|\varepsilon_0(t_k)| + \bar{\beta}_{\mathcal{U}}(t_k)) e^{-a_m (t - t_k)} + \\ &+ \quad a(k) e^{-\lambda_c t} + 2\|(\bar{\beta}_{\mathcal{U}})_t\| \,, \end{aligned} \tag{30}$$

$\forall t \in [t_k, t_M)$, with λ_c in (22) and $a(k)$ is any positive monotonically increasing unbounded sequence. The *monitoring function for* $n^* > 1$ φ_m is defined by

$$\varphi_m(t) \quad := \quad \varphi_k(t) \,, \forall t \in [t_k, t_{k+1}) \subset [0, t_M) \,. \tag{31}$$

Note that φ_m is discontinuous in t. The switching time t_k for u from u^- to u^+ (or u^+ to u^-) is well-defined by:

$$t_{k+1} := \begin{cases} \min\{t > t_k : |\varepsilon_0(t)| = \varphi_k(t)\}, & \text{if it exists} \,, \\ t_M, & \text{otherwise} \,, \end{cases} \tag{32}$$

where $k \geq 1$, $t_0 := 0$ and $t_1 := \bar{t}_e$. For convenience, $\varphi_0 := 0, \forall t \in [t_0, t_1)$. The following proposition follows directly from the definition of the monitoring function φ_m, in (31).

Proposition 1: Let $k \geq 1$ be the largest switching index of the monitoring function (31), such that $t_k \in [0, t_M)$, then the auxiliary error $\varepsilon_0(t)$ is bounded by

$$|\varepsilon_0(t)| \leq \varphi_m(t), \quad \forall t \in [t_1, t_M) \,. \tag{33}$$

C. Stability Results ($n^* > 1$)

The following proposition assures boundness of the error system during the switching of the control direction.

Proposition 2: Assume that (A1)-(A6) hold. Consider the complete error system (4), (18) and (19), with state z defined in (24). Let k be the largest switching index of the monitoring function, given in (31), such that $t_k < t < t_M$. Then, $\forall R_0 > 0$ such that $|z(0)| \leq R_0$, there exists a sufficiently small $\tau_2 > 0$ (that depends on R_0 and k) such that $\forall \tau \in (0, \tau_2]$, the complete error system is bounded by

$$|z(t)| \leq k_{z0}|z(0)| + k_a \sum_{i=1}^{k} a(i) + \Psi(\tau) + k_\tau k\tau + \mathcal{O}(\tau)\,, \quad (34)$$

$\forall t \in [0, t_M)$, where $\Psi \in \mathcal{K}$ and k_{z0}, k_a, k_τ are positive constants.

Proof: See Appendix. ■

The main stability and convergence result can be stated in the following theorem (for the linear lead filter compensation):

Theorem 2: Assume that (A1)–(A6) hold, the modulation function is given by (9) and $ML(s) = k_m/(s + a_m)$ with $(k_m, a_m > 0)$. Then, for sufficiently small $\tau > 0$, the switchings of the control sign, driven by the monitoring function (31), stop after a finite number of switchings and the complete error system (4), (18) and (19), with state z defined in (24) is semi-globally asymptotically stable with respect to a compact set and ultimately exponentially convergent to a residual set of order $\mathcal{O}(\tau)$, both sets being independent of the initial conditions. If the nonlinearity in the system satisfies a global Lipschitz condition, the semi-global stability properties become global.

Proof: Propositions 1 and 2 are used to prove that the modification of the control direction by the switching rule (18), based on the monitoring function (31), will stop after a finite number of switching. Moreover, during this phase the complete state of error system remains uniformly bounded. After that, the proof follows from [14, Theorem 2]. See the Appendix for a complete proof. ■

1) Chattering Avoidance and Exact Tracking : In Figure 1, the block L_{red} represents the "exact lead filter" which implements the operator $L(s)$ by using the RED. The RED algorithm to compute the first and second derivatives of a given signal e_0 is given by:

$$\begin{aligned}
\dot{\eta}_0 &= v_0, \\
v_0 &= -\lambda_0 \left|\eta_0 - e_0\right|^{\frac{2}{3}} \operatorname{sgn}(\eta_0 - e_0) + \eta_1, \\
\dot{\eta}_1 &= v_1, \\
v_1 &= -\lambda_1 \left|\eta_1 - v_0\right|^{\frac{1}{2}} \operatorname{sgn}(\eta_1 - v_0) + \eta_2, \\
\dot{\eta}_2 &= -\lambda_2 \operatorname{sgn}(\eta_2 - v_1)\,,
\end{aligned} \quad (35)$$

which provides $\eta_0(t) \to e_0(t)$, $\eta_1(t) \to \dot{e}_0(t)$ e $\eta_2(t) \to \ddot{e}_0(t)$. Higher order differentiators can be found in [18], [16]. We can therefore state exact tracking result:

Corollary 2: With the hybrid lead filter, all results of the Theorem 2 hold and moreover, exact tracking is achieved in finite time or at least exponentially. Moreover, the control sign switchings stops at the correct sign.

Proof: The resulting global/semi-global stability and convergence of the modified scheme can be easily proved since, as already remarked above, by proper design of the switching law, the hybrid lead filter only introduces a disturbance β_α which is norm-bounded by a design constant of order $\mathcal{O}(\tau)$, *modulo* decaying exponential terms, which can be embedded in e_F^0 (22). This constant bound can be simply added to $\bar{\beta}_{\mathcal{U}}$ given in (29). The monitoring function can be redefined in an appropriate way in order to monitor the perturbed auxiliary signal $\tilde{\varepsilon}_0$. The control sign switchings will stop in finite time and the exact differentiator takes over since the error systems enters the residual set. Then, the system becomes exactly a relative one case. Thus the *reverse dynamics argument* can be applied (see proof of Theorem 1) to show that the control sign switchings stop at the correct sign. ■

D. Robustness to Measurement Noise

To date, there is no SMC which is immune to measurement noise. This is more critical with output feedback schemes when differentiation underlies the existing controllers, including those based on HGOs and the one presented here. On the other hand, it should be emphasized that the parameter τ can be chosen not so small, so that high frequency noise can be filtered out. The role of the linear lead filter is only to guarantee stability at large while the RED takes care of the tracking precision.

The hybrid lead filter scheme has been experimentally verified in [17]. Experimental results with the proposed scheme (to be presented in a future paper) have shown that robustness to noise is, to a certain extent, acceptable.

V. Simulation Results

This section presents an illustrative simulation example which highlights the performance of the proposed control scheme for a nonlinear plant with relative degree $n^* = 3$.

Example 1: Consider an open-loop unstable plant with transfer function given by: $G_p(s) = \frac{1}{(s+2)(s+1)(s-1)}$, being controlled by the VS-MRAC controller of Figure 1 and under the action of a nonlinear input disturbance $d_e(y, t) = y^2 + sqw(5t)$, where *sqw* denotes a unit square wave. The reference model is $M(s) = \frac{4}{(s+2)^3}$ and the linear lead filter is given in (17) with $L(s) = (s + 2)^2$ and $\tau = 0.01$. The monitoring function is obtained from (31) with $a(k) = k+1$ and $\bar{\lambda}_c = 0.5$. The plant initial conditions are $y(0) = 2$, $\dot{y}(0) = 0$ and $\ddot{y}(0) = 2$ and the feedback is positive at $t = 0$ (wrong control direction).

Figure 2 corresponds to a simulation result when the reference signal is a sinusoid of amplitude 1 and frequency 1 rad/s. The convergence of the plant output signal to the model reference output is clear. Figure 3 (a) shows that just one switching in the control sign was need (first jump of φ_m when it meets $\tilde{\varepsilon}_0$). After that, the control direction is correctly identified and the auxiliary error $\tilde{\varepsilon}_0$, as well as the tracking error, vanish in finite time. Note that the second discontinuous-like change of φ_m is not due to a change between u^+ and u^-. It is due to the $\|(\bar{\beta}_{\mathcal{U}})_t\|$ term in (30).

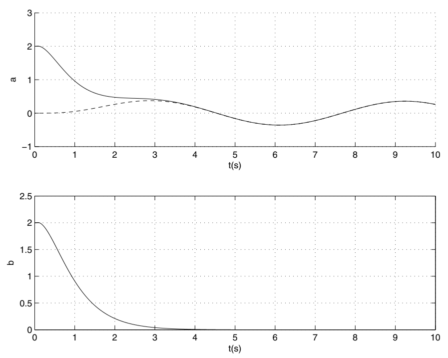

Fig. 2. Simulation results (Example 1). Simulated (a) plant output y (line), reference model output y_m (dash) and (b) the plant tracking error e_0.

Figure 3 (b) shows the changes between the linear ($\alpha = 1$) and the nonlinear ($\alpha = 0$) lead filters. It is clear that the nonlinear lead filter is ultimately chosen by the switching strategy of the hybrid lead filter. This is also indicated by the convergence to zero (in finite time) of both the plant tracking error (e_0) and the filtered error $\tilde{\varepsilon}_0$.

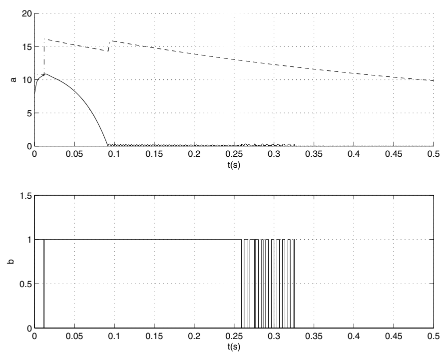

Fig. 3. Simulation results (Example 1); (a) monitoring function φ_m (dash) and the auxiliary error $\tilde{\varepsilon}_0$ (line); (b) switching function α: $\alpha = 1$ (linear lead filter) and $\alpha = 0$ (nonlinear lead filter).

VI. Conclusions

An output-feedback model-reference sliding mode controller was developed for a class of nonlinear uncertain systems with unknown control direction. The proposed controller is an extension of the VS-MRAC controller, introduced in [1], to nonlinear systems with arbitrary relative degree. The controller requires two switching schemes: one to adjust the control direction and another to compensate the relative degree in such a way that global/semi-global stability holds and also exact tracking is obtained. The control direction adjustment was based on a monitoring function constructed from input and output error signals. The relative degree compensation was based on the switching between a linear lead filter and a locally exact differentiators based on 2-sliding modes. The resulting controller leads to global or semi-global asymptotic stability with respect to some compact set and ultimate exponential or finite time convergence of the tracking error to zero. The controller has led to quite reasonable transient behavior in our simulations in contrast to the Nussbaum gain approach.

VII. Acknowledgments

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Appendix

Please refer to [**www.coep.ufrj.br/˜liu/vss06**] for the detailed proofs of stability.

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