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Title: Smooth Sliding Control of Van der Pol Oscillators with a Single Input: Application to Micro-Thermal-Fluid Cooling Systems
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B. *Proof of Theorem 1*

The proof is carried out in two parts: before and after sliding mode takes place.

Part A: Analysis During the Reaching Phase

From Proposition 1, there exists a finite time $t_s \in [0, t_M)$ such that, $\forall t \in [t_s, t_M)$, sliding mode occurs, i.e., the sliding variable $\tilde{\sigma}(t)$ becomes identically null. However, at this point, it does not assure directly that finite time escape is avoided in the closed-loop system signals. Finite-time escape avoidance is proven in what follows. From Proposition 1, one has that the *δ-reachability condition*

$$\dot{\epsilon}_0 \epsilon_0 \;=\; \frac{1}{2}\frac{d}{dt}\left[\epsilon_0^2(t)\right] \leq -\delta_\varrho |\epsilon_0(t)| \,, \tag{16}$$

holds. Then, integrating (16) from t_0 to $t \in [t_0, t_M)$, with $t \leq T_s$ and $T_s := t_0 + |\epsilon_0(t_0)|/\delta_\varrho$, it follows that

$$|\epsilon_0(t)| \;\leq\; -\delta(t - t_0) + |\epsilon_0(t_0)| \leq |\epsilon_0(t_0)| \,,$$

$\forall t \in [t_0, t_M)$, and $t \leq T_s$. It is clear that $\epsilon_0(t_0) = 0$ implies sliding mode at the manifold $\epsilon_0(t) \equiv 0$, starting from the beginning, i.e., $\forall t \in [t_0, t_M)$, since $T_s = t_0 + |\epsilon_0(t_0)|/\delta = t_0$ and the *δ-reachability condition* (16) is satisfied. In this case, finite-time escape cannot occur before sliding mode takes place. Thus, from now on, assume that $\epsilon_0(t_0) \neq 0$.

Assuming that t_M is finite, then there exists a finite t^* ($t_0 < t_M < t^*$) such that some close-loop signal escapes at $t = t^*$. Moreover, aiming to prove that finite-time escape cannot occur before sliding mode takes place, assume that $\epsilon_0(t) \neq 0$, $\forall t \in [t_0, t^*]$.

Due to the unboundedness observability property of the closed-loop system, finite-time escape can occur if and only if the output $\sigma_0 = \dot{e}_0 + \lambda_1 e_0$ escapes in finite-time. In addition, since the *δ-reachability condition* holds, then ϵ_0 is uniformly norm bound in the time interval $[t_0, t^*]$.

Then, $\hat{e}_0(t) = \bar{e}_0(t) - \epsilon_0(t)$ must also escapes at $t = t^*$ and $\lim_{t \to t^*} |\hat{e}_0(t)| = \infty$. However, at this point, $\hat{e}_0$ can escape to infinity oscillating around zero (and switching sign), or monotonically with a fixed sign. Both cases do not occur. Indeed, since $\hat{e}_0 = k^{nom} M(s) L(s)[u_0 - u_0^{av}] = k^{nom} M(s) L(s)[1 - F_{av}^{-1}(\tau s)]u_0 = k^{nom} M(s) L(s)[1 - F_{av}^{-1}(\tau s)]\varrho \text{sgn}(\epsilon_0)$, where $\text{sgn}(\epsilon_0(t))$ is fixed in the time interval $[t_0, t^*]$. Thus, $\hat{e}_0$ cannot escape in finite time. Consequently, $\bar{e}_0$ and e_0 cannot escape in finite time. Finally, one can conclude that sliding mode occurs before any closed-loop signal escapes in finite time. However, finite-time escape is not precluded after sliding mode takes place. To complete the proof, we will evoke the Small Gain Theorem.

Part B: Analysis in Sliding Mode

From **Part (a)**, there exists a finite time $t_s \in [0, t_M)$ such that, $\forall t \in [t_s, t_M)$, sliding mode occurs, i.e., the sliding variable $\epsilon_0(t) = 0$ becomes identically null.

Since $\epsilon_0 := \bar{e}_0 - \hat{e}_0 = 0$, thus $\bar{e}_0 = \hat{e}_0 = k^{nom} M L(s)[u_0 - u_0^{av}]$. Therefore, one can write $\frac{1}{F(\tau_F s)}\sigma_0 = k^{nom} M L(s)[F_{av}(\tau s) - 1]u_0^{av*}$, where

$$u_0^{av*} = \frac{1}{k^{nom} M(s) L(s)(F_{av}(\tau s) - 1) F(\tau_F s)}\sigma_0 \,,$$

compose the *synthesized DSSC* law $u_r^* = u^{nom} - u_0^{av*}$, i.e., the equivalent control law during sliding mode. Note that, the σ_0-dynamics can be written as

$$\sigma_0 = k^* M(s) L(s)[-u_0^{av} + d_\sigma] \,, \tag{17}$$

where $d_\sigma := u^{nom} - u^* + W_d(s)d$. Thus, with $u_0^{av} = u_0^{av*}$, one can further write

$$\left[1 + \frac{k^*}{k^{nom}(F_{av}(\tau s) - 1)F(\tau_F s)}\right]\sigma_0 = k^* M(s) L(s) d_\sigma \,,$$

or, equivalently,

$$\sigma_0 = P(s) d_\sigma \,,$$

where

$$P(s) = \frac{k^* k^{nom}(F_{av}(\tau s) - 1)F(\tau_F s) M(s) L(s)}{k^{nom}(F_{av}(\tau s) - 1)F(\tau_F s) + k^*} \,.$$

In the Appendix III-C, one can subsequently verify that: (i) the ideal matching control u^* can be represented as a filtered version of σ_0 (plus feedforward terms); (ii) since the nominal control can also be represented as a filtered version of σ_0 (plus feedforward terms), so is the disturbance $d_\sigma := u^{nom} - u^* + W_d(s)d$, plus the filtered disturbance $W_d(s)d$, and (iii) the transfer function $P(s)$ is of order $\mathcal{O}(\tau + \tau_F)$, i.e., $\|P(s)\|_\infty = \mathcal{O}(\tau + \tau_F)$. So, one can write (see Appendix III-C)

$$\sigma_0 = P_\sigma(s)\sigma_0 + P(s)W_d(s)d + \bar{u}_m \,, \tag{18}$$

where $P_\sigma(s)$ and $P(s)W_d(s)$ are strictly stable transfer functions of order $\mathcal{O}(\tau + \tau_F)$ and $\bar{u}_m$ is feedforward signal that is norm bounded by a constant of order $\mathcal{O}(\tau + \tau_F)$.

Now, given a ball of radius $R > 0$, such that $x_1^2 + x_2^2 < R^2$, then one has $|x_1| < R$ and $|x_2| < R$. Let $K_{R1}(R)$ and $K_{R2}(R)$ positive constants depending on R, such that

$$|x_1| < R < (2K_{R1})^{1/3} \,, \quad |x_2| < R < 2K_{R2} \,,$$

which leads to

$$|x_1|^4 < 2K_{R1}|x_1| \,, \quad |x_2|^2 < 2K_{R2}|x_2| \,.$$

Moreover, the following inequalities hold

$$|x_1|^2|x_2| \leq (|x_1|^4 + |x_2|^2)/2 \leq K_R(|x_1| + |x_2|) \,, \tag{19}$$

with $K_R = K_{R1} + K_{R2}$. Now, from (**??**) and reminding that $|g(x_1)| \leq k_{g1}x_1^2 + k_{g2}$, one has $|d| \leq k_{g1}|x_1|^2|x_2| + k_{g2}|x_2|$ and

$$|d| \leq (k_{g1}K_R)|x_1| + (k_{g1}K_R + k_{g2})|x_2| \,.$$

In addition, since $x_1 = e_0 + y_m$ and $x_2 = \dot{e}_0 + \dot{y}_m$, one can further write

$$x_1 = \frac{1}{L(s)}\sigma_0 + y_m \,, \quad x_2 = \frac{s}{L(s)}\sigma_0 + \dot{y}_m \,,$$

leading to the inequalities

$$|x_1| \leq \frac{1}{\lambda_1}\|\sigma_0\|_\infty + \|y_m\|_\infty + \pi_1 \,,$$

and

$$|x_2| \leq \|\sigma_0\|_\infty + \|\dot{y}_m\|_\infty + \pi_2 \,,$$

since $L(s) = s + \lambda_1$, where π_1 and π_2 are exponentially decaying terms due to initial conditions.

Therefore, one can conclude that

$$|d| \leq \left[\left(k_{g1}K_R + k_{g2} + \frac{k_{g1}K_R}{\lambda_1}\right)\|\sigma_0\|_\infty + d_m + \pi\right] \,,$$

where $d_m := (k_{g1}K_R)\|y_m\|_\infty + (k_{g1}K_R + k_{g2})\|\dot{y}_m\|_\infty$ and $\pi := (k_{g1}K_R)\pi_1 + (k_{g1}K_R + k_{g2})\pi_2$. The proof follows by applying the Small Gain Theorem to (18), leading, subsequently, to the semi-global convergence of σ_0 and e_0 to a residual set of order $\mathcal{O}(\tau + \tau_F)$, while all closed-loop signals remain uniformly norm bounded, so that finite-time escape is avoided $(t_M \to \infty)$. ■

C. Development of (18)

Let us rewrite the nominal control law as

$$u^{nom} = C_n(s)e_0 + u_m^n = \frac{C_n(s)}{L(s)}\sigma_0 + u_m^n \,,$$

where the relationship $\sigma_0 = L(s)e_0$ was used. It must be highlighted that the nominal control is not regarded as a disturbance and can be disregarded when the plant uncertainty is large. Reminding that $u_r^* = u^{nom} - u_0^{av*}$ and

$$u_0^{av*} = \frac{1}{k^{nom}M(s)L(s)(F_{av}(\tau s) - 1)F(\tau_F s)}\sigma_0 \,,$$

one can write

$$u_r^* = \frac{C_n(s)}{L(s)}\sigma_0 + u_m^n - C_{av}(s)\sigma_0 \,,$$

where $C_{av}(s) = \frac{1}{k^{nom}M(s)L(s)(F_{av}(\tau s)-1)F(\tau_F s)}$. In the MRC approach the ideal control u^* is parameterized as $u^* = \theta^{*T}\omega$, where $\theta^* = \begin{bmatrix} \theta_m^* & \theta_1^{*T} & \theta_2^{*T} & \theta_y^{*T} \end{bmatrix}^T \in \mathbb{R}^{2n}$ is the ideal parameter vector, $\omega = \begin{bmatrix} y_m & v_1^T & v_2^T & \mathcal{Y} \end{bmatrix}^T \in \mathbb{R}^{2n}$ is the regressor vector and $v_1 \in \mathbb{R}^{n-1}$ and $v_2 \in \mathbb{R}^{n-1}$ are the input and output state variable filters and n is the order of the plant. The input and output state variable filters are given by strictly stable transfers functions $F_1(s)$ and $F_2(s)$ such that

$$\theta_1^{*T} v_1 = F_1(s) u_r \,, \quad \theta_2^{*T} v_2 = F_2(s) \mathcal{Y} \,.$$

So, one can write

$$u^* = \theta_m^* y_m + F_1(s) u_r + F_2(s) \mathcal{Y} + \theta_y^* \mathcal{Y} \,,$$

or, equivalently,

$$u^* = u_m^* + F_1(s) u_r + \frac{F_2(s)}{L(s)} \sigma_0 + \frac{\theta_y^*}{L(s)} \sigma_0 \,,$$

where

$$u_m^* = \theta_m^* y_m + F_2(s) y_m + \theta_y^* y_m \,.$$

Now, considering the synthesized control law $u_r = u_r^*$, one has

$$\begin{aligned} u^* \;&=\; \left[\frac{F_1(s) C_n(s)}{L(s)} - F_1(s) C_{av}(s) + \frac{F_2(s) + \theta_y^*}{L(s)} \right] \sigma_0 \\ &+\; u_m^* + F_1(s) u_m^n \,, \end{aligned} \tag{20}$$

Reminding that $\sigma_0 = P(s) d_\sigma$ and $d_\sigma := u^{nom} - u^* + W_d(s) d$ one has

$$\sigma_0 = \frac{P(s) C_n(s)}{L(s)} \sigma_0 + P(s) u_m^n + P(s) u^* + P(s) W_d(s) d \,.$$

Hence, one can write

$$\sigma_0 = P_\sigma(s) \sigma_0 + P(s) W_d(s) d + \bar{u}_m \,,$$

where $P_\sigma(s) := \frac{F_1(s)P(s)C_n(s)}{L(s)} - F_1(s) P(s) C_{av}(s) + \frac{P(s)(F_2(s)+\theta_y^*)}{L(s)} + \frac{P(s)C_n(s)}{L(s)}$ and $\bar{u}_m := P(s) u_m^* + P(s)(F_1(s)+1) u_m^n$. Moreover,

$$P(s) W_d(s) = P(s) \bar{W}_d(s) L(s)(s + a_m)/k^* \,,$$

since $W_d(s) = [k^* M(s)]^{-1} \bar{W}_d(s)$.

One can verify that $P(s) W_d(s)$ and $P_\sigma(s)$ are transfer functions of order $\mathcal{O}(\tau + \tau_F)$. To verify that and since we are considering the relative degree two case to simplify the presentation, recall that $F(\tau_F s) = \tau_F s + 1$ and write

$$P(s) = \frac{k^* k^{nom} \tau (\tau_F s + 1) s}{(k^{nom} \tau (\tau_F s + 1) s + k^*)(s + a_m)} \,.$$

Let us consider $\tau_F = \alpha k^{nom} \tau$, with $\alpha > 0$. This is not restrictive, and assures that τ_F must be small when τ is made small. The lead-filter time constant must be of the order of the averaging filter time constant.

Then, one has that the polinomial

$$(k^{nom} \tau (\tau_F s + 1) s + k^*) = (k^{nom} \tau)^2 \alpha s^2 + k^{nom} \tau s + k^* \,,$$

has real roots

$$\frac{-k^{nom} \tau \pm \sqrt{(k^{nom}\tau)^2 - 4 (k^{nom}\tau)^2 k^* \alpha}}{2 (k^{nom}\tau)^2 \alpha} \,,$$

or,

$$-\frac{p_i}{\tau_F} = \frac{-1 + (-1)^i \sqrt{1 - 4k^* \alpha}}{2\tau_F} \,, \quad (i = 0, 1) \,,$$

provide that $\alpha < 1/(4k^*)$. Now, for each fixed $0 < \alpha < 1/(4k^*)$ one has that p_0 and p_1 are independent of τ_F (or τ), so that one can rewrite $P(s)$ in the forms

$$P(s) = \frac{k^* (s + 1/\tau_F) s}{(s + p_0/\tau_F)(s + p_1/\tau_F)(s + \bar{a}_m/\tau_F)} \,,$$

or, equivalently,

$$P(s') = \tau_F \left[\frac{k^* (s' + 1) s'}{(s' + p_0)(s' + p_1)(s' + \bar{a}_m)} \right] \,,$$

where $k^{'} := \left\| \frac{k^*(s^{'}+1)s^{'}}{(s^{'}+p_0)(s^{'}+p_1)(s^{'}+\bar{a}_m)} \right\|_\infty = \mathcal{O}(1)$, $s^{'} = \tau_F s$, and $a_m = \bar{a}_m/\tau_F$. Thus, one can write

$$\frac{\|P(s)\|_\infty}{\tau+\tau_F} \leq \frac{\tau_F}{\tau+\tau_F}k^{'} \leq \frac{\alpha k^{nom}}{1+\alpha k^{nom}}k^{'}\,,$$

and conclude that

$$\|P(s)\|_\infty = \mathcal{O}(\tau+\tau_F)\,.$$

Moreover, one can subsequently conclude that all transfer functions composing $P_\sigma(s)$, $P_\sigma(s)$ and $P(s)W_d(s)$ are of order $\mathcal{O}(\tau+\tau_F)$. In addition, one can subsequently conclude that $P(s)(F_1(s)+1)$ is of order $\mathcal{O}(\tau+\tau_F)$ and $\bar{u}_m$ is norm bounded by a constant of order $\mathcal{O}(\tau+\tau_F)$. ■