Potential Function Formation Control of Nonholonomic Mobile Robots with Curvature Constraints

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Abstract

In this paper two formation control strategies for nonholonomic mobile robots with curvature constraints are proposed. The strategies are based on a saturated potential function which makes possible the design of decentralized formation control scheme and avoids agents collisions considering its dimensions. Provided that the communication graph is always connected, the control strategies guarantee the formation will achieve a configuration that minimizes the potential function.

Key words: Nonholonomic; Formation stability; Decentralized control; Curvature constraints; Nonlinear systems.

1 Introduction

Formation control of multiple agents has received significant attention from the control community due to its wide variety of applications. Among several formation control strategies used, can be mentioned the behavior-based \cite{17}, consensus \cite{16}, leader-following \cite{19}, group coordination using passivity \cite{1}, virtual structures \cite{2,10} and potential function with virtual leader \cite{11}.

For mobile robots formation control, the main objective is to control each robot using neighbors information in a decentralized control strategy. In this framework, most of the existing results deal with holonomic mobile robots \cite{15,14,20,18}. However, in practical applications, mobile robots have to satisfy nonholonomic constraint. The control design for nonholonomic systems is quite involved, mainly due to the Brockett’s condition \cite{4}. Therefore, for agents with nonholonomic constraint, the formation control problem becomes more challenging.

In \cite{19}, stability properties of formation of mobile agents based on leader-following are investigated. In \cite{7} a combined kinematic/torque control law is proposed for leader-follower based formation control based on backstepping. In \cite{13,12}, a decentralized control scheme which achieves dynamic formation control and collision avoidance for a group of nonholonomic robots with kinematic model is proposed. The collision avoidance strategy is based on locally defined potential functions which can take different shapes and only require each agent to detect other objects in its neighborhood. In \cite{8}, a decentralized feedback control of a group of nonholonomic dynamic systems with uncertainty is considered. The control scheme is based on consensus, graph theory, and backstepping techniques. In \cite{9}, an adaptive formation control for nonholonomic mobile robots with unknown dynamic parameters is proposed. The control scheme is based on a saturated artificial potential function which allows a decentralized formation control design including collision avoidance.

In \cite{5,6} a geometric approach for the stabilization of a hierarchical formation of unicycle with velocity and curvature constraint is proposed, where the leader-following strategy is used. However, collision avoidance is not considered. Furthermore, a drawback of leader-following strategies is that it depends heavily on the leader for achieving the goal and over-reliance on a single agent in the formation may be undesirable, especially in adverse conditions.

In this paper, a formation control law of nonholonomic mobile robots based on potential function strategy is proposed. The trajectory curvature of each robot is bounded. The control law is developed using a satu-

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rated potential function that makes possible the design of decentralized formation control laws considering agents collisions avoidance. The proposed control law guarantees that, if the communication graph is always connected, the formation will achieve a configuration that minimizes the potential function.

2 Problem Formulation

Consider a team of \( N \) front-wheel drive car. For \( i = 1, \cdots, N \), the \( i \)-th robot kinematic model is given by:

\[
\begin{bmatrix}
    \dot{x}_i \\
    \dot{y}_i \\
    \dot{\theta}_i \\
    \dot{\delta}_i
\end{bmatrix} =
\begin{bmatrix}
    \cos(\theta_i) & 0 \\
    \sin(\theta_i) & 0 \\
    \frac{L_i}{v_i} \tan(\delta_i) & 0 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    v_i \\
    \omega_i
\end{bmatrix}
\]

(1)

where \( r_i = [x_i, y_i, \theta_i, \delta_i]^T \) is the cartesian coordinates, \( \theta_i \) is the robot orientation with respect to the horizontal, \( \delta_i \) is the steering angle of front wheels with respect to the robot body (see Fig. 1), \( v_i \) is the linear velocity of the rear wheels, \( \omega_i \) is the angular velocity of the steering wheels and \( L_i \) is the wheelbase, here, for simplicity \( L_i = L \) \( \forall i \). The rear wheels are considered aligned with the car.

![Kinematic model of an automobile](image)

Fig. 1. Kinematic model of an automobile

The topology of information exchange among robots is described by a graph [3]. A graph \( \mathcal{G} \) consists of a vertex set \( \mathcal{V} \) and an edge set \( \mathcal{E} \), i.e. \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \). Thus, the \( N \) mobile robots are represented as \( N \) vertices in \( \mathcal{V} \), where the communication among robots is described by the edge set \( \mathcal{E} \). An orientation of a graph \( \mathcal{G} \) (directed graph) is the assignment of a direction to each edge, so that the edge \((i, j) \in \mathcal{E}\) is now an arc from vertex \( i \) to vertex \( j \). A graph is undirected if the edges have no orientation \(((i, j) = (j, i) \in \mathcal{E})\). Let \( \mathcal{N}_i \) be a collection of neighbors of robot \( i \), i.e., a set of indexes of robots whose states are available to robot \( i \). The available information for the controller of robot \( i \) is only the states of robot \( i \) and robot \( j \) for \( j \in \mathcal{N}_i \). A path of length \( r \) from robot \( i \) to robot \( j \) is a sequence of \( r + 1 \) distinct vertices starting with \( i \) and ending with \( j \) such that consecutive vertices are neighbors. If there is a path between any two vertices of a graph \( \mathcal{G} \), then \( \mathcal{G} \) is said to be connected. A graph is a tree if is connected and without cycles, that is any two vertices are connected by exactly one simple path.

For an directed graph (digraph), the incidence matrix \( B(\mathcal{G}) \) is the matrix whose rows and columns are indexed by the vertices and edges of \( \mathcal{G} \) respectively, such that the \( i, j \) entry of \( B(\mathcal{G}) \) is equal to 1 if edge \( j \) is incoming to vertex \( i \), -1 if edge \( j \) is outgoing from vertex \( i \), and 0 otherwise.

In this paper the information flow between neighbors is assumed to be bidirectional.

Here, the main objective is to design a decentralized formation control for vehicles with trajectory curvature constraints, i.e.

\[|\delta_i| \leq \delta_{\text{max}} < \frac{\pi}{2},\]

The formation control is based on potential functions of relative position \(|r_{ij}|\) between agents \( i \) and \( j \), where \( r_{ij} = r_i - r_j \).

First, define

\[V = \sum_{i=1}^{N} V_i; \quad V_i = \sum_{j \in \mathcal{N}_i, j \neq i} V_{ij}(|r_{ij}|)\]

(2)

where \( V_{ij}(|r_{ij}|) > 0 \) is defined as [18]:

**Definition 1** The saturated potential function \( V_{ij} \) is a differentiable, nonnegative function of \(|r_{ij}| \in (c, \infty)\), such that

1. \( V_{ij}(|r_{ij}|) \to \infty \) as \(|r_{ij}| \to c\), where \( c > 0 \).
2. \( V_{ij} \) attains its unique minimum when agents \( i \) and \( j \) are located at a desired relative position \( r_d > c \).
3. \( V_{ij}(|r_{ij}|) = V_{ij}(R_s) \) if \(|r_{ij}| \geq R_s\), where \( R_s > r_d \).

Figure 2 shows an example of a saturated potential function.

The desired formation is achieved when the minimum of \( V \) is reached. Thus, artificial forces can be created to minimize \( V_i \) for all \( i \) (and consequently \( V \)) and avoid collision between robots. These artificial forces can be defined as

\[f_{ij} = -\nabla r_{ij} V_{ij}(|r_{ij}|)\]

(3)

where \( \nabla r_{ij} V_{ij}(|r_{ij}|) = \left[ \frac{\partial V_{ij}}{\partial x_j}, \frac{\partial V_{ij}}{\partial y_j} \right] \) is the gradient of \( V_{ij} \).

Note that \( f_{ij} \) is an attraction force for \(|r_{ij}| > r_d\), or a repulsion force for \(|r_{ij}| < r_d \). Force \( f_{ij} \) is null for \(|r_{ij}| = r_d \) and \(|r_{ij}| > R_s \).
V_{ij}(\|r_{ij}\|)

Fig. 2. Saturated potential function

Thus, to reach the minimum of $V_i$, each robot has to move in the direction of resulting artificial force

$$f_i = \sum_{j \in \mathcal{N}_i, j \neq i} f_{ij} = -\sum_{j \in \mathcal{N}_i, j \neq i} \nabla_r V_i(\|r_{ij}\|) = -\nabla_r V_i$$

(4)

From definition 1, $V_{ij}(\|r_{ij}\|) \to \infty$ as $\|r_{ij}\| \to \infty$, thus robot dimensions can be taken into account for the formation control. On the other hand, $R_s$ defines the neighborhood region of each robot. Then, agent $j$ belongs to $\mathcal{N}_i$ if and only if $\|r_{ij}\| < R_s$. Thus, it possible to consider a region of limited communication about each agent, which allows a decentralized control design.

It is important to stress that the saturated potential function generates a switching interconnection topology. Nevertheless, in this paper the communication graph is always assumed to be connected.

$$\delta_i \to \delta_{id} \text{ as } t \to \infty.$$ (a)

$$v_i > 0 \text{ if } |\gamma_i| \leq \frac{\pi}{2}.$$ (b)

$$v_i < 0 \text{ if } |\gamma_i| > \frac{\pi}{2}.$$ (c)

Note that the objective of criteria (a) is the robot $i$ to track the dashed line in Fig. 3. On the other hand, the objective of criterias (b) and (c) is the velocity direction of each vehicle $i$ to be the same direction of its resulting artificial force $f_i$ if $|\gamma_i| = 0$ or $|\gamma_i| = \pi$. In this paper, two control strategies are proposed. In what follows, the first of them is shown.

### 3 Projection Control Law

Defining the unit vector $R_i^T = [\cos(\theta_i), \sin(\theta_i)]$ (see Fig. 3), the following control law is proposed

$$v_i = K_{id} f_i R_i$$

(9)

$$\omega_i = -K (\delta_i - \delta_{id})$$

(10)

As $|R_i| = 1$ and from (4) and Fig. 3, note that $v_i = K_{id} \nabla_r V_i \cos(\gamma_i)$. As $\cos(\gamma_i) < 0$ for $|\gamma_i| > \pi/2$ and $\cos(\gamma_i) \geq 0$ for $|\gamma_i| \leq \pi/2$, then the criterias (2) and (3) are satisfied.
Defining

\[ C_\gamma = \text{diag}(\cos(\gamma_1), \cos(\gamma_2), \cdots, \cos(\gamma_N)), \]  

the following Theorem shows that the formation achieves a configuration that minimizes \( V_i \) if \( T_f(C_\gamma) \neq 0 \).

**Theorem 1** Consider \( N \) mobile robots with kinematic model (1), each robot steered by control law (9)–(10) and exchanging information through a switching communication graph. For

1. \( |r_{ij}(0)| > c \) for all \( i,j \),
2. an always connected communication graph,
3. formation states initial condition \( z(0) \) starting from a set \( \mathcal{D} = \{ z : W(z) \leq W_0 \} \) where \( W_0 > 0 \), \( W \) is a Lyapunov function defined as

\[
2W(z) = \sum_{i=1}^{N} \left( (\delta_i - \delta_{id})^2 + \alpha_1 V_i \right), \tag{12}
\]

\[
z^T = [(\delta_1 - \delta_{id}) \cdots (\delta_N - \delta_{id}) \ r^T_1 \cdots r^T_N]
\]

and \( r^T_i = [x_i \ y_i] \), then the robots achieve a formation such that \( \nabla_{r_i} V_i R_i = 0 \ \forall i \). Moreover, if \( T_f(C_\gamma) \neq 0 \), then (2) is minimized when the formation is achieved.

**Proof:**

Consider the Lyapunov candidate function (12). The derivative of \( W \) with respect to time is given by

\[
\dot{W} = \sum_{i=1}^{N} \left[ (\delta_i - \delta_{id})(\dot{\delta}_i - \dot{\delta}_{id}) + \frac{\alpha_1}{2} \sum_{j=1}^{N} \left( \frac{\partial V_i}{\partial x_i} \dot{x}_i + \frac{\partial V_i}{\partial y_i} \dot{y}_i \right) \right]. \tag{13}
\]

Due to \( V_i \) being symmetric with respect to \( r_{ij} = r_i - r_j \) and the fact that \( r_{ij} = -r_{ji} \) (undirected graph), one has that

\[
\frac{\partial V_i}{\partial r_i} = \frac{\partial V_j}{\partial r_j} = \frac{\partial V_{ij}}{\partial r_i}, \tag{14}
\]

and therefore it follows that:

\[
\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \frac{\partial V_i}{\partial x_i} \dot{x}_i + \frac{\partial V_i}{\partial y_i} \dot{y}_i \right) = \sum_{i=1}^{N} \nabla_{r_i} V_i \dot{r}_i \tag{15}
\]

Then, considering (15), the derivative of \( W \) along trajectory of the system (1)(9)–(10) is given by

\[
\dot{W} = \sum_{i=1}^{N} \left[ -K (\delta_i - \delta_{id})^2 + (\delta_i - \delta_{id}) \dot{\delta}_{id} - \alpha_1 K_v (\nabla_{r_i} V_i R_i)^2 \right]. \tag{16}
\]

On the other hand, considering (6)–(7), one has that

\[
\dot{\delta}_{id} = \frac{\cos(\gamma_i)}{\rho_{\min} (1 + L^2 K_{id}^2)} \dot{\gamma}_i \tag{17}
\]

\[
\dot{\gamma}_i = \frac{K_v}{L} \tan(\delta_i) \nabla_{r_i} V_i R_i - K_v L_i \sum_{j=1}^{N} L_{2ij} \nabla_{r_j} V_j R_j \tag{18}
\]

\[
L_{1i} = \left( \frac{\partial V_i}{\partial x_i} \right)^2 + \left( \frac{\partial V_i}{\partial y_i} \right)^2 \tag{19}
\]

\[
L_{2ij} = \frac{\partial^2 V_i}{\partial y_j \partial x_j} \sin(\theta_j) + \frac{\partial^2 V_i}{\partial y_j \partial x_j} \cos(\theta_j) \tag{20}
\]

Then, replacing (18), (17) in (16) and after some algebraic manipulation, one has that

\[
\dot{W} = \sum_{i=1}^{N} \left[ -K (\delta_i - \delta_{id})^2 - \alpha_1 K_v (\nabla_{r_i} V_i R_i)^2 \right. \right.
\]

\[
- \frac{(\delta_i - \delta_{id}) \cos(\gamma_i)}{\rho_{\min} (1 + L^2 K_{id}^2)} L_i \sum_{j=1}^{N} (L_{2ij} K_v \nabla_{r_j} V_j R_j) \right. \right.
\]

\[
+ \frac{(\delta_i - \delta_{id}) \cos(\gamma_i)}{\rho_{\min} (1 + L^2 K_{id}^2)} L_i \nabla_{r_i} V_i \nabla_{r_i} V_i R_i \right]. \]

Defining the following errors

\[
e_{\delta}^T = [(\delta_1 - \delta_{id}) \cdots (\delta_N - \delta_{id})] \tag{21}
\]

\[
T_{\Delta} = [\nabla_{r_1} V_1 R_1 \cdots \nabla_{r_N} V_N R_N] \tag{22}
\]

and

\[
J_1 = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{1N} \\ J_{12} & J_{12} & \cdots & J_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ J_{1N} & J_{1N} & \cdots & J_{1NN} \end{bmatrix}, \tag{24}
\]

with

\[
J_{1ij} = \begin{cases} \frac{\cos(\gamma_i) (\tan(\delta_i) - L_i L_{1ij} L_{2ij})}{\rho_{\min} L (1 + L^2 K_{id}^2)} & \text{for } i = j \\ \frac{\cos(\gamma_i) L_{1i} L_{2ij}}{\rho_{\min} (1 + L^2 K_{id}^2)} & \text{for } i \neq j \end{cases} \tag{25}
\]
one has that
\[ \dot{W} \leq -K_\delta \epsilon_\delta + K_\epsilon \epsilon_\delta^T J_1 e_{\Delta t} - \alpha_1 K_\epsilon \epsilon_{\Delta t} e_{\Delta t}. \]  
(26)

Therefore
\[
\dot{W} \leq -e^T \begin{bmatrix} K I_{N \times N} & -0.5 K_\epsilon J_1 \\ -0.5 K_\epsilon J_1 & \alpha_1 K_\epsilon I_{N \times N} \end{bmatrix} e,
\]
where \( e^T = [\epsilon_\delta^T \epsilon_{\Delta t}^T]. \)

Assuming that \( z \in \mathcal{D}, \) there exist a finite constant such that \( |J_1| < J_1. \) Hence, for \( \dot{W} < 0 \) the Shur complement of \( C \) should satisfy
\[
\left( K - \frac{K_\epsilon J_1^2}{4\alpha_1} \right) > 0 \tag{27}
\]
Then, (27) will be satisfy if
\[
K > \frac{K_\epsilon J_1^2}{4\alpha_1}. \tag{28}
\]

Therefore if (28) hold, the set \( \mathcal{D} \) is invariant (so that the assumed uniform bounds of \( \|J_1\| \) hold), \( |e| \to 0 \) as \( t \to \infty \) and so does \( |e_{\Delta t}|. \) Then, from (23), the formation achieves a configuration such that \( \nabla \dot{r}_i V_i R_i \to 0 \) \( \forall i. \)

Note that \( \epsilon_{\Delta t} \to 0 \) does not imply \( |\epsilon_{\Delta t}| \to 0. \) However, since \( \nabla \dot{r}_i V_i R_i = [\nabla \dot{r}_i V_i \cos(\gamma_i)] e_{\Delta t} = C e_{\Delta t}. \) Thus for \( \text{Tr}(C) \neq 0, \) one has that \( \epsilon_{\Delta t} \to 0 \) as \( t \to \infty. \) Therefore \( |\nabla \dot{r}_i V_i| \to 0 \) \( \forall i \) as \( t \to \infty \) and the formation achieves a configuration that minimizes \( V. \)

Remark 1 It was not possible to proof the Theorem 1 considering the Lyapunov function (12) as a function of \( z \) and \( \theta \) where \( \theta^T = [\theta_1 \ldots \theta_N]. \) Then, although from (1), (4) and (9), \( \nabla \dot{r}_i V_i R_i \to 0 \) \( \Rightarrow \dot{\theta}_i \to 0, \) it does not imply that \( |\theta| \) is bounded. Although the simulation results shows \( |\theta| \) bounded, its mathematical proof is yet an open problem. However, if \( \epsilon_i \) be defined such that for \( \nabla \dot{r}_i V_i R_i < \epsilon_i, \) \( \epsilon_i \) is setted zero then, from (1), \( \theta_i \) will be bounded. In this case, the formation achieves a set \( \mathcal{D}_i = \{z \mid \nabla \dot{r}_i V_i R_i < \epsilon_i\}, \) which is a neighborhood of order \( O(\epsilon_i) \) of minimum of \( V \) if \( \text{Tr}(C) \neq 0. \)

Note that, if \( \text{Tr}(C) \neq 0, \) the \( N \) robots will be not simultaneously orthogonal to its resulting artificial force \( f_i. \) However, though unlikely, \( \text{Tr}(C) = 0 \) is possible mathematically. The Fig. 4 shows an example where \( \text{Tr}(C) = 0. \) This configuration is an equilibrium point of formation (\( z \) constant) independently of \( V_i \) (not only on minimum of \( V_i) \).

Fig. 4. Formation configuration with \( \text{Tr}(C_i) = 0 \)

4 Bias Control Law

In this section, a control law is proposed where the hypothesis \( \text{Tr}(C) \neq 0 \) used in Theorem 1 is not necessary. Thus, as following, a bias term \( \epsilon [f_i] \) is introduced in (9) and (10),

\[
v_i = \begin{cases} \epsilon |f_i| + K_\epsilon f_i R_i & \text{if } |\gamma_i| \leq \frac{\pi}{2} \\ -\epsilon |f_i| + K_\epsilon f_i R_i & \text{otherwise} \end{cases} \tag{29}
\]
\[
\omega_i = -K (\delta_i - \delta_{id}) \tag{30}
\]

Note that the criteria (2) and (3) are satisfied. Note too that, using the control laws (29) and (30), the formation configuration shown in Fig. 4 is no longer an equilibrium point of formation. However, \( |\nabla \dot{r}_i V_i| = 0 \) is yet an equilibrium point.

The following Theorem shows that, for the control laws (29) and (30), \( \delta_i \to \delta_{id} \) \( \forall i \) as \( t \to \infty \) and the \( N \) robots achieve a formation such that \( \nabla \dot{r}_i V_i R_i \to 0 \) \( \forall i \) as \( t \to \infty. \)

Theorem 2 Consider \( N \) mobile robots with kinematic model (1), each robot steered by control law (29)–(30) and exchanging information through a switching communication graph. For

(1) \( |r_{ij}(0)| > c \) for all \( i, j, \)
(2) an always connected communication graph,
(3) formation states initial condition \( z(0) \) starting from a set \( \mathcal{D} = \{z \mid W(z) \leq W_0\} \) where \( W_0 > 0 \) and \( W \) is a Lyapunov function defined by (12),

then \( \delta_i \to \delta_{id} \) \( \forall i \) as \( t \to \infty \) and the \( N \) robots achieve a formation such that \( \nabla \dot{r}_i V_i R_i \to 0 \) \( \forall i \) as \( t \to \infty. \)
Proof: Since (29) is a switched control law, the following two cases have to be considered:

Case 1: $|\gamma_i| \leq \frac{\pi}{2}$

Consider the Lyapunov candidate function (12). Then, considering (15), (17) and using the control laws (29) and (30), we have

$$\dot{W} = \sum_{i=1}^{N} \left\{ -K \left( \delta_i - \delta_{id} \right)^2 + \frac{(\delta_i - \delta_{id}) \cos(\gamma_i) \gamma_i}{\rho_{\min}(1 + L^2 K_{id}^2)} - \alpha_1 \delta_i \left| \nabla_r V_i R_i \right| \left| \nabla_r V_i \right| - \alpha_1 K_v \left( \nabla_r V_i R_i \right)^2 \right\}$$  \hspace{1cm} (31)

where, differentiating (6),

$$\dot{\gamma}_i = \frac{K_v}{L} \tan(\delta_i) \left| \nabla_r V_i \right| R_i + K_v L_{1,i} \sum_{j=1}^{N} L_{21,j} \left| \nabla_r V_j R_j \right|$$

and $L_{1,i}$ and $L_{21,j}$ are as in (19) and (20) respectively. Since $\left| R_i \right| = 1$,

$$f_i R_i = -\nabla_r V_i R_i = -\nabla_r V_i \cos(\gamma_i)$$  \hspace{1cm} (33)

(see Figure 3). Thus, for $|\gamma_i| \leq \frac{\pi}{2}$, $\left| \nabla_r V_i R_i \right| = \left| \nabla_r V_i \cos(\gamma_i) \right|$ and $\left| \nabla_r V_i R_i \right| = -\nabla_r V_i R_i$. Then, after some algebraic manipulation, one has that

$$\dot{W} \leq \sum_{i=1}^{N} \left\{ -K \left( \delta_i - \delta_{id} \right)^2 - \alpha_1 \epsilon \left| \nabla_r V_i R_i \right| \left| \nabla_r V_i \right| + \frac{(\delta_i - \delta_{id}) \cos(\gamma_i) \gamma_i}{\rho_{\min}(1 + L^2 K_{id}^2)} - \frac{(\delta_i - \delta_{id}) \cos(\gamma_i) \tan(\delta_i) \epsilon}{\rho_{\min}(1 + L^2 K_{id}^2)} \left| \nabla_r V_i \right| - \alpha_1 K_v \left( \nabla_r V_i R_i \right)^2 + \frac{(\delta_i - \delta_{id}) \cos(\gamma_i) \gamma_i}{\rho_{\min}(1 + L^2 K_{id}^2)} K_v L_{1,i} \sum_{j=1}^{N} \left| L_{21,j} \left| \nabla_r V_j R_j \right| \right| \right\}$$  \hspace{1cm} (34)

Noting that $-\left| \nabla_r V_i R_i \right|^2 \geq -\left| \nabla_r V_i \right| \left| \nabla_r V_i R_i \right|$ and us-
Assuming that \( z \in \mathcal{D} \), there exist finite constants \( J_1, J_2, J_3 \) such that \( |J_1| < J_1, |J_2| < J_2 \) and \( |J_3| < J_3 \). Therefore

\[
\dot{W} \leq -e^T \begin{bmatrix} K & 0.5 J_4 \\ -0.5 J_4 & \alpha_1 (\epsilon + K_3) \end{bmatrix} e
\]

(42)

where \( e^T = [|e_\phi| \ |e_\Delta|] \) and \( J_4 = K e_\phi J_1 + \epsilon (\beta J_2 + J_3) \).

Hence, for \( \dot{W} < 0 \) the Shur complement of \( C \) should satisfy

\[
K - \frac{J_2^2}{4 \alpha_1 (\epsilon + K_3)} > 0
\]

(43)

Then, (43) will be satisfy if

\[
K > \frac{J_2^2}{4 \alpha_1 (\epsilon + K_3)}
\]

(44)

Therefore if (44) hold, the set \( \mathcal{D} \) is invariant (so that the assumed uniform bounds of \( |J_1|, |J_2| \) and \( |J_3| \) hold), \( |e| \) \( \rightarrow \infty \) as \( t \rightarrow \infty \) and do so \( |e_\Delta| \) and \( |e_\phi| \). Then, from (21) and (23), one can conclude that, for all \( i, \delta_i \rightarrow \delta_{id} \) and \( V_{ri} R_i \rightarrow 0 \) as \( t \rightarrow 0 \).

**Case 2:** \( |\gamma_i| > \frac{\pi}{2} \)

Consider the Lyapunov candidate function (12). Then, using the same mathematical development as in case (1), one has that

\[
\dot{W} = \sum_{i=1}^{N} \left[ -K (\delta_i - \delta_{id})^2 + \frac{(\delta_i - \delta_{id}) \cos(\gamma_i) \gamma_i}{\rho_{min}} (1 + Lq K_{id})^2 \right. \\
- \alpha_1 e \|V_{ri} R_i\| \|V_{ri} R_i\| - \alpha_1 e \|V_{ri} R_i\| \|V_{ri} R_i\| + \alpha_1 K e (\nabla_{ri} V_{ri} R_i)^2 \right]
\]

(45)

where only \( \gamma_i \) is changed to

\[
\gamma_i = \frac{K}{L} \tan(\delta_i) |V_{ri} R_i| - K e L q \sum_{j=1}^{N} L_{2ij} |V_{ri} R_i|
\]

\[
+ \frac{e}{L} \tan(\delta_i) |V_{ri} R_i| + e \|V_{ri} R_i\| \left( \sum_{j=1}^{N} e L_{2ij} \|V_{ri} R_i\| \left( \sum_{j=1}^{N} e L_{2ij} \right) \right)
\]

(46)

Using (33), it can be concluded that, in this case, \( V_{ri} R_i = |V_{ri} R_i| \). Then,

\[
\dot{W} = -K e \|V_{ri} R_i\| - \alpha_1 (\epsilon + K_2 \epsilon_\phi \epsilon_\Delta) - \epsilon_\phi J_2 e \|V_{ri} R_i\| + \epsilon_\phi J_2 e \|V_{ri} R_i\| + \epsilon_\phi J_2 e \|V_{ri} R_i\| - \epsilon_\phi J_2 e \|V_{ri} R_i\|.
\]

(47)

Thus, \( \dot{W} \) is described by (40) and the proof follows the same lines as case (1).

\[\blacksquare\]

It is important to note that in contrast of proof of Theorem 1, the hypothesis \( \text{Tr}(C_2) \neq 0 \) is not considered in proof of Theorem 2. This because, for the control law (29)-(30), the condition \( \text{Tr}(C_2) = 0 \) is not an equilibrium point of closed loop system. Indeed, as is shown in following Lemma, even if \( \text{Tr}(C_2) = 0 \), the \( N \) robots will achieve a formation in a neighborhood \( \mathcal{D}_2 \) of order \( O(\rho_{min}) \) of minimum of \( V \) where, in \( \mathcal{D}_2, r_{im} \leq \rho_{min} \forall i \) and \( r_{im} \) is the distance of each robot to minimum of its potential function \( V_i \).

**Corollary 1** Consider a formation with \( N \) mobile robots with kinematic model (1), each robot steered by control law (29)-(30). Then, this formation achieves a neighborhood \( \mathcal{D}_2 \) of order \( O(\rho_{min}) \) of minimum of \( V \) where, in \( \mathcal{D}_2, r_{im} \leq \rho_{min} \forall i \).

**Proof:** For \( \text{Tr}(C_2) \neq 0 \) or \( v_{ri} \rightarrow 0 \) \( \forall i \) as \( t \rightarrow \infty \), it can be concluded that \( |V_{ri} R_i| \rightarrow 0 \) \( \forall i \). Then \( V \) is minimized and the neighborhood \( \mathcal{D}_2 \) is achieved. However, the critical case is \( \text{Tr}(C_2) = 0 \), where \( \gamma_i = \frac{\pi}{2} \) \( \forall i \). Then, two situations are analyzed:

(a) \( \gamma_i \neq 0 \) for some \( i \).

(b) \( \gamma_i = 0 \forall i \).

For (a), the formation configuration with \( \text{Tr}(C_2) = 0 \) is not maintained. Then, using the same argument as in case \( \text{Tr}(C_2) \neq 0 \), it can be concluded that \( V \) is minimized and the neighborhood \( \mathcal{D}_2 \) is achieved. For (b), since \( \gamma_i = \frac{\pi}{2} \) \( \forall i \) then \( V_{ri} R_i = 0 \) \( \forall i \) (see Figure 3). Thus, differentiating (6) with respect to time, we have

\[
\dot{\gamma_i} = \dot{\beta_i} - \frac{1}{L} \tan(\delta_i) v_i = 0,
\]

where \( v_i = \epsilon \|f_i\| \) and \( \dot{\beta_i} \) is the derivative with respect to time of

\[
\beta_i = \arctan 2 \left( -\frac{\partial V_i}{\partial y} \frac{\partial V_i}{\partial x} \right).
\]

Hence,

\[
\dot{\beta_i} = \frac{1}{L} \tan(\delta_i) v_i.
\]

As \( \delta_i \rightarrow \delta_{id} \) as \( t \rightarrow \infty \), \( \dot{\beta_i} \rightarrow \frac{1}{L} \tan(\delta_{id}) v_i \). However, from (5) and (7), \( \tan(\delta_{id}) = \frac{e}{\rho_{min}} \). Thus \( v_i \rightarrow \dot{\beta_i} \rho_{min} \). Then, the \( N \) robots will be over circular trajectories with radius \( \rho_{min} \) with angular velocity \( \dot{\beta} \). Note that this trajectory has the maximum curvature \( K_{imax} \) where \( \|K_{imax}\| = \frac{1}{\rho_{min}} \) (see (8)). Therefore, while \( r_{im} > \rho_{min} \), \( K_{imax} \) > \( \frac{1}{\rho_{min}} \), and the robot \( i \) will be approaching of \( r_{im} \). Then, the \( N \) robots achieve a formation in a neighborhood \( \mathcal{D}_2 \) of order \( O(\rho_{min}) \) of minimum of \( V \) where, in \( \mathcal{D}_2, r_{im} \leq \rho_{min} \forall i \).

\[\blacksquare\]
5 Simulation Results

This section presents the simulation results for a formation with six and four vehicle. The goal is to compare the performance of the two proposed control strategies, described by (9), (10) and (29), (30). Then, consider that each vehicle of formation has wheelbase \(L = 3\) and the potential function among vehicles is given by

\[
V_{ij}(\|r_{ij}\|) = \begin{cases} 
J(\|r_{ij}\|) & \text{for } |r_{ij}| < R_s \\
J(R_s) & \text{for } |r_{ij}| \geq R_s 
\end{cases}
\]  
(48)

where

\[
J(|r_{ij}|) = \log (|r_{ij} - c|) + \frac{a_2}{|r_{ij} - c|} - a_1 (|r_{ij} - c|)
\]  
(49)

and

\[
a_1 = \frac{1}{r_d - c + R_s}
\]

\[
a_2 = \frac{R_s(r_d - c)}{r_d - c + R_s}
\]

\(c = 3, R_s = 300\) and \(r_d = 10\). Note that \(J_{ij}(\|r_{ij}\|)\) has a global minimum at \(|r_{ij}| = r_d\).

The controller gain are \(K_0 = 5, K = 10\) and \(\epsilon = 0.01\) and the angle of front wheels is bounded by \(|\delta_i| \leq \pi/3, \forall i\). In what following, the simulation results are shown. The performance difference between the proposed control laws becomes more evident in simulation with four robots.

5.1 Formation with Six Vehicle

The initial conditions are: \(\delta_i = 0 \forall i\) and

\[
r_1(0) = [0, 100]^T, \quad \theta_1(0) = \pi/2
\]

\[
r_2(0) = [30, 100]^T, \quad \theta_2(0) = -\pi/3
\]

\[
r_3(0) = [80, 0]^T, \quad \theta_3(0) = \pi/2
\]

\[
r_4(0) = [-80, 0]^T, \quad \theta_4(0) = -\pi
\]

\[
r_5(0) = [0, -100]^T, \quad \theta_5(0) = \pi
\]

\[
r_6(0) = [80, -100]^T, \quad \theta_6(0) = -\pi/2.
\]

Here, the simulation results shown are obtained using the projection control law (9), (10). Because \(\epsilon < < K, K_v\), the simulation results using the bias control law (29), (30) did not make a noticeable difference and do not shown.

The Figure 5 shows the robots trajectories until the formation to be achieved.

5.2 Formation with Four Vehicles

\(|r_{ij}| \neq 0\) at all time, the simulation results confirm that the collision is avoided.

From Figure 7 one can observe that \(-\pi/3 \leq \delta_i \leq \pi/3\). Then, the trajectories curvature are limited. The Figure 8 shows the orientation of robots.

The initial conditions are: \(\delta_i = \pi/3 \forall i\) and

\[
r_1(0) = [100, 100]^T, \quad \theta_1(0) = 3\pi/4
\]

\[
r_2(0) = [-100, 100]^T, \quad \theta_2(0) = -3\pi/4
\]

\[
r_3(0) = [-100, -100]^T, \quad \theta_3(0) = -\pi/4
\]

\[
r_4(0) = [100, -100]^T, \quad \theta_4(0) = \pi/4.
\]
This initial conditions was chosen to \( f_i R_i = 0 \ \forall i \).
Then, for \( t = 0 \), the four vehicles are simultaneously orthogonal to its resulting artificial force \( f_i (\gamma_i = \pi/2) \).
The Figures 9 and 10 show the trajectories and the relative positions among robots using the projection control law (9),(10). Since \( f_i R_i = 0 \), the vehicles stand still and a formation is not achieved.

The simulation results using the bias control law (29), (30) are shown in Figures 11 and 12, witch show that the formation is achieved. Then, the difference of performance between the projection control law and bias control law can be easily observed.

6 Conclusions

The design of a decentralized formation controller for nonholonomic mobile robots with curvature constraint was presented. The control scheme is based on potential functions which is able to avoid agents collision. Simulation results shown that if the communication graph is connected, a formation that minimize the potential function will be achieved. Future directions include the problem of formation control of dynamic nonholonomic mobile robots, trajectory tracking problem, where the formation has to follow a leader and the formation control of the underactuated mechanical systems.
Fig. 11. Robots trajectories

Fig. 12. Relative positions among robots

References


