Sliding Mode Output Tracking of Uncertain Nonlinear Systems with Unknown Control Direction.

Tiago Roux Oliveira, Alessandro Jacoud Peixoto and Liu Hsu

Abstract—An output feedback sliding mode controller was recently introduced for linear uncertain systems with unknown control direction based on monitoring functions. Nonlinear systems were also considered but restricted to output dependent nonlinearities. Global or semi-global exact output tracking was obtained. Here, generalization is achieved to include nonlinearities depending on unmeasured states and to deal with multivariable systems. Furthermore, a new monitoring function to cope with control direction uncertainty is proposed which can lead to significantly improved tracking transient behavior.

Keywords: uncertain nonlinear systems, sliding mode control, output feedback, unknown control direction, exact tracking, multivariable systems.

I. INTRODUCTION

An output feedback tracking sliding mode control (SMC) for single-input-single-output (SISO) uncertain linear plants with relative degree one and unknown control direction (i.e., the sign of the high frequency gain) was introduced in [1]. In lieu of the Nussbaum gain [2], the control sign was adjusted through a monitoring function. Related results for arbitrary relative degree are in [3] (linear plants) and [4] (linear plants with output dependent nonlinearity). In this paper, the nonlinear disturbances are allowed to be state dependent and unmatched. As in [3], [4], relative degree compensation and asymptotic exact tracking are achieved by means of a hybrid lead filter [5] based on robust exact feedback linearization [8].

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1 As in [10], assumption (A2) can be relaxed to: $k_p := \frac{L_B p}{L_f p^{n-1}} (C_p x_p)$ and $|k_p - C_p p^{n-1} B_p| \leq \varepsilon$, for sufficiently small $\varepsilon > 0$.

II. PROBLEM FORMULATION

Notation: Euclidean norm of a vector $x$ and induced norm of a matrix $A$: $||x||$ and $||A||$, respectively; $\mathcal{L}_\infty$ norm of the signal $x(t) \in \mathbb{R}^n$, from initial time $t_0$: $\|x(t)\| := \sup_{t \in \mathbb{R}^n} \|x(t)\|$; for $t_0 = 0$, $\|x_0\|$ is adopted; “$d/dt$” according to the context; the output of a linear system with transfer function $H(s)$ and input $u$ is written $H(s)u$; pure convolution $h(t) * u(t)$; $H(s)u, h(t)$ being the impulse response from $H(s)$; classes $K, K_\infty$ functions are defined as usual [8, pp. 144]; Lie derivative of a function $h$ along a vector field $f$ is denoted by $L_f h$, as in [8, pp. 510].

Consider a SISO nonlinear uncertain plant described by

$$\dot{x}_p = f_p(x_p, t) + B_p u_p, \quad y = C_p x_p,$$

where $f_p(x_p, t) = A_p x_p + \phi(x_p, t)$, $x_p \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the measured output and $\phi : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is regarded as an uncertain state dependent nonlinear disturbance, possibly unmatched. The triple $(A_p, B_p, C_p)$ is in the canonical controllable form with uncertain constant matrices $A_p$ (lower companion form) and $C_p = [c_{p1} c_{p2} \ldots c_{p(n-n^*)+1} 0 \ldots 0]$.

All uncertain parameters belong to some compact set $\Omega_p$ such that the necessary uncertainty bounds to be defined later are available for design. In $\Omega_p$ we assume that: (i) $\phi$ is locally Lipschitz in $x_p$ ($\forall x_p$) and piecewise continuous in $t$ ($\forall t$); (ii) $(A_p, B_p, C_p)$ is minimum-phase, observable, has known order $n$ and known relative degree $n^*$, as usual in Model Reference Adaptive Control (MRAC) [9]. Our main additional assumptions are:

(A1) A global diffeomorphism $T : x_p \rightarrow (\eta^T, \xi^T)^T, \eta \in \mathbb{R}^{n-n^*}, \xi = [y_1, y_2, \ldots, y^{n^*-1}]^T$ transforms (1) into the normal form, where $\eta$ represents the zero dynamics state vector [8, pp. 516].

(A2) The nonlinear plant high frequency gain (HFG) $k_p := C_p A_p^{-1} B_p = c_{p(n-n^*+1)}$ is constant $1$.

(A3) $\phi$ satisfies $|\phi(x_p, t)| \leq k \phi x_p + \phi(y, t), \forall x_p, t$, where $k \geq 0$ is a known scalar and $\phi$ is a known function such that $\phi(y, t) \leq \Psi_c (|y|) + k_\phi$, where $\Psi_c \in K_\infty$ and $k_\phi > 0$ is constant.

Assumption (A1) allows us to reduce $\phi$ to a nonlinear input disturbance and (A2) assumes a constant HFG $k_p$ but otherwise uncertain in norm and sign. Hence the control direction is uncertain. System (1) with $\phi$ triangular in the unmeasured states $\phi(x_p) = [\phi_1(x_{p1}) \phi_2(x_{p2}) \ldots \phi_n(x_{pn})]^T$ satisfies (A1)–(A2). According to (A3), no particular growth condition is imposed on $\phi$, e.g., $\phi(y) = y^2$ which grows quadratically. Thus, finite-time escape is not precluded a priori. The maximal time interval of definition of a given solution is given by $[0,T_M]$, where $T_M$ may be finite or
infinite. Filippov’s definition for the solution of discontinuous differential equations is assumed throughout the paper.

The aim is to achieve global or semi-global stability properties in the sense of uniform signal boundedness and asymptotic output tracking, i.e., the output tracking error \( c_0(t) = y(t) - y_m(t) \) should tend to zero (exact tracking). The desired trajectory \( y_m(t) \) is assumed to be generated by the following reference model:

\[
y_m = M(s)r = \frac{k_m}{L(s)(s + a_m)} r, \quad k_m, a_m > 0,
\]

where \( L(s) = s^N + a_{N-1}s^{N-1} + \ldots + a_0 (N := n^* - 1) \) is a Hurwitz polynomial and the reference signal \( r(t) \) is assumed piecewise continuous and uniformly bounded.

III. OUTPUT ERROR EQUATION

Following the usual MRAC approach, with \( \phi \) regarded as a disturbance, one can write:

\[
\dot{X} = A_rX + b_r r + b_{r\phi} (u - \theta^T \omega(t)) + B_{\phi} \phi, \quad y = h^T X, \tag{3}
\]

where \( X^T := [x_{\phi}^T \omega_1^T \omega_2^T] \), with \( \omega_1 \) and \( \omega_2 \) being the states of the input/output filters which form the regressor vector \( \omega = [\omega_1^T \omega_2^T y] r^T \) [9]. In (3), \( B^T = [1 \ 0 \ 0] \) and \( (A, b_r, b_{r\phi}) \) is a non-minimal realization of \( M(s) \). The ideal parameter vector \( \theta^* \) is such that the closed-loop transfer function (with \( u = \theta^T \omega \) and \( \phi \equiv 0 \)), from \( r \) to \( y \), matches \( M(s) \) exactly. The matching conditions require \( \rho^* = k_p / k_m \). Although, \( \theta^* \) is unknown, it is elementwise bounded by a known constant vector \( \theta^0 \).

From (3), (A1) and (A2), obtaining the successive output time derivatives of order up to \( n^* - 1 \) and using the Markov parameters for representing \( M(s) \), one can write

\[
e_0 = \rho^* M(s) [u - u^*], \quad u^* := \theta^T \omega(t) - d_\phi, \tag{4}
\]

where \( u^* \) is the model matching control in the presence of \( \phi \), which is treated as an equivalent nonlinear input disturbance

\[
d_\phi := W_{n^*} \phi_1 \phi_{n^*-1} + \ldots + W_1 \phi_1 W_0 \phi + \hat{W}_\phi (s) * \phi, \tag{5}
\]

with \( \hat{W}_\phi (s) \) being strictly proper and BIBO stable and \( W_i \in \mathbb{R}^{n \times n} \) being appropriate constant vectors [10, pp. 203]. Note that, (A1)–(A2) guarantees that \( u \) does not appear in \( d_\phi \) (5).

We further assume that:

(A4) The terms \( \phi, \phi_1, \ldots, \phi_{n^*-2} \) and their partial derivatives are continuous w.r.t. their arguments.

In order to account for the complete error state, consider the state vector \( X_e = X - X_m \) of the stable non-minimal realization of \( M(s) \) in (4) with matrices \( (A, b_r, b_{r\phi}) \), where \( X_m = A_r X_m + b_r r^T / \theta^* - d_\phi + B_{\phi} \phi \) and \( d_\phi \) cancels the effect of \( B_{\phi} \phi \) in \( y_m = h^T X_m \).

IV. NONLINEAR INPUT DISTURBANCE NORM BOUND

From (A4), one can find \( \phi_0 \in K \) and a constant \( k_\phi > 0 \) such that \( |d_\phi| \leq \Phi_\phi(|x_p|) + \hat{W}_\phi (s) |\phi| + k_\phi \). Considering (A3), it is possible to find \( k_\phi^* > 0 \) such that, for \( k_\phi \in [0, k_\phi^*] \), \( B_{\phi} \phi \) preserves the stability of \( A_r \). Thus, applying [12, Lemma 3] to the solution of (3) a norm bound for \( X, x_p \) can be obtained through first order approximation filters (FOAFs).

Indeed, one has \( |x_p(t)| \leq \hat{x}_p(t) + \hat{\pi}(t) \), where

\[
\hat{x}_p(t) = \frac{1}{s + \lambda_x} [c_1 \varphi(y, t) + c_2 |\omega(t)|], \tag{6}
\]

with \( c_1, c_2, \lambda_x > 0 \) being appropriate constants. The exponentially decaying term \( \hat{\pi} \) accounts for initial conditions, see [12]. From (A3) and (6), the following upper bound holds

\[
|\phi(x_p, t)| \leq k_x \hat{x}_p(t) + \varphi(y, t), \tag{7}
\]

modulo \( \hat{\pi} \) term and one can write \( |d_\phi| \leq \hat{d}_\phi + \hat{\pi}(t) \), where

\[
\hat{d}_\phi(t) := \Psi_\phi(\hat{x}_p(t)) + \frac{c_\phi}{s + \gamma_\phi} [k_x \hat{x}_p(t) + \varphi(y, t)] + k_\phi, \tag{8}
\]

\( \hat{\pi} \) is a vanishing term and \( c_\phi, \gamma_\phi > 0 \) are constants.

V. OUTPUT FEEDBACK SLIDING MODE CONTROLLER

As in [3], [4], a hybrid lead filter (Fig.1) named Global Robust Exact Differentiator (GRED) [5] will be used to compensate the excess of relative degree and reduce the problem to the \( n^* = 1 \) case. The GRED provides a surrogate for the non-causal operator \( L(s) \) in (2). The GRED combines linear \( (L/F, F(\tau s + 1)^N, \tau > 0) \) and nonlinear \( (L_{red}) \) lead compensation, the latter being constructed with REDs [6]. A convex combination based on a continuous switching law \( \alpha \) [5], results in ultimate exact compensation of the relative degree while assuring global or semi-global stability properties of the closed loop system.

A. Control Law

According to Fig. 1, the control signal \( u \) is defined as:

\[
\begin{align*}
\begin{cases}
u^+ &= -f(t) \text{sgn}(\hat{e}_0) \quad t \in T^+ \\
u^- &= f(t) \text{sgn}(\hat{e}_0) \quad t \in T^- 
\end{cases}
\end{align*}
\]

where \( \hat{e}_0 = (1 - \alpha) e_0 + \alpha e_0 \). A monitoring function \( \varphi_m \) for \( \hat{e}_0 \) is used to decide when \( u \) should be switched from \( u^+ \) to \( u^- \) and vice versa, based on the detection of any wrong estimate of \( \text{sgn}(k_\phi) \). The sets \( T^+ \) and \( T^- \) satisfy \( T^+ \cup T^- = [0, t_M) \) and \( T^+ \cap T^- = 0 \), and both have the form \( [t_k, t_{k+1}) \cup \cdots \cup [t_i, t_{i+1}) \). Here, \( t_k \) or \( t_i \) denote switching times for \( u \).

B. Modulation Function

As seen from the stability analysis, the modulation function \( f(t) \) must satisfy \( f \geq |u^*| \). This holds, modulo exponentially decaying or vanishing terms, with

\[
f(t) = |\theta^T \omega(t)| + |\hat{d}_\phi(t)| + \delta, \tag{10}
\]

where \( \delta > 0 \) is an arbitrary constant. The vector \( \theta^T \) is such that \( \theta^T = \max(|\theta^T|, |\theta^T|) \), where \( \theta \) is the model matching vector with respect to an unstable reference model \( M^T(s) = \frac{k_0}{s(s - a_m)}, k_m, a_m > 0 \). This will guarantee that, with wrong control direction, the system would become unstable and ensures that the control direction can be correctly found with the monitoring function.
The monitoring function was developed based on the function \( \psi_k^{od}(t) := (|\varepsilon_0(t_k)| + |\beta(t_k)|)e^{-a_m(t-t_k)} + 2\gamma(t) \).

**B. Conservative Monitoring Function**

From (13), one possible choice for \( \gamma(t) \) is given by \( \gamma(t) := \left[ R_u e^{\lambda c t} e^{-\lambda c t} + \|\beta(t)\| \right] \), where \( \lambda_c = \min\{a_m, \theta\} \). In [3], [4], from (15) and this choice for \( \gamma \), the *original conservative* monitoring function was developed based on the function \( \psi_k^{od}(t) := (|\varepsilon_0(t_k)| + |\beta(t_k)|)e^{-a_m(t-t_k)} + a(k)e^{-\lambda c t + 2}\|\beta(t)\| \), defined \( \forall t \in [t_k, t_M] \). Since \( R_u \) and \( \beta \) are not available, the positive monotonically increasing sequence \( a(k) \) and the signal

\[
\bar{\beta} = 2\rho^* W_{\beta}(s, \tau) * f(t), \quad \rho^* \geq \rho^*,
\]

(\( \bar{\beta} \geq |\beta| \)) are introduced. The operator \( W_{\beta}(s, \tau) \) is chosen, through partial fraction expansion of the operator in (12), as a sum of FOAFs, such that the induced \( L_\infty \) norm of \( W_{\beta}(s, \tau) \) is of order \( O(\tau) \).

**C. New Monitoring Function with Forgetting**

Here, we propose a new \( \gamma \) which tends to be arbitrarily small if \( |\beta(t)| \) tends to a sufficiently small value as \( \tau \to \infty \), thus avoiding large transients whenever the control direction changes [7]. To this end, let

\[
\gamma(t) := \left( R_u e^{\lambda c t} e^{-\lambda c t} + f_d(t) \right), \quad f_d(t) := \left\{ \begin{array}{l}
\|\beta_d(t)\|, \\
\|\beta(t)\|, \\
\|\beta(t)\| e^{-\sigma(t-t_j)}, \end{array} \right. \text{for } T_j < t \leq T_{j+1},
\]

where \( j \geq 1 \) and \( \sigma \leq a_m \). The time instants \( T_j, j \in \{1, 2, \ldots, j^*\} \) and \( T_j, j \in \{0, 1, \ldots, j^*\} \) are defined by

\[
T_{j+1} := \min\{t > T_j : |\beta(t)| \leq S f_d(t)\}, \quad T_{j} := \min\{t > T_{j-1} : |\beta(t)| \leq \beta(t)\},
\]

where \( \bar{T}_0 = 0, 0 < \mu < 1 \) and \( j^* \) (or \( j^* \)) is the first index such that the minima in (19) or (20) do not exist. The decaying rate \( \sigma \) acts like a *forgetting factor* which provides a vanishing property for \( f_d(t) \) in (18) (see Fig. 2).

The new monitoring function \( \varphi_m(t) \) is given by

\[
\varphi_m(t) := \varphi_k(t), \forall t \in [t_k, t_{k+1}) \subset [0, t_M),
\]

(21)

where \( \varphi_k(t) \) is obtained from \( \varphi_k^{od}(t) \) by replacing \( \|\beta(t)\| \) by \( f_d(t) \) as defined above. The switching time \( t_k \) from \( u^+ \) to \( u^- \) or \( u^+ \) to \( u^- \) is defined by

\[
t_{k+1} := \left\{ \begin{array}{l}
\min\{t > t_k : |\varepsilon_0(t_k) = \varphi_k(t)\}, \quad \text{if exists}, \\
t_M, \quad \text{otherwise},
\end{array} \right.
\]

(22)

where \( k \in \{1, 2, \ldots\} \), \( t_0 := 0 \) and \( t_1 := t_e \). For convenience, \( \varphi_0 := 0, \forall t \in [t_0, t_1) \). The following inequality is directly obtained from definition (21)

\[
|\varepsilon_0(t)| \leq \varphi_m(t), \quad \forall t \in [t_1, t_M).
\]

**C. Equivalent Structure for the Hybrid Lead Filter**

According to [5, Lemma 2], with an appropriate switching function \( \alpha \), the GRED is equivalent to a linear lead filter perturbed by a uniformly bounded output measurement disturbance \( \bar{\beta} \), of order \( O(\tau) \). In order to simplify the analysis, we postpone its consideration to Subsection VIII-A. Thus, in what follows, we assume \( \varepsilon_0 = \varepsilon_0 \).

**VI. LEAD FILTER ERROR EQUATION**

Henceforth, let \( \tau \in (0, \tilde{\tau}] \), where \( \tilde{\tau} < 1 \) is some sufficiently small constant. As in high gain observers, lead filters inevitably generate peaking as \( \tau \to 0 \) [8]. Through a convenient realization [3], [4], a peaking free state vector \( x_f \) is obtained for \( L/F \) (Fig. 1). Peaking appears only at the output \( \varepsilon_0 \) of the linear lead filter. The \( \text{sgn}(\cdot) \) function in the control law (9) blocks the transmission of the peaking to the plant. From (4), since \( \varepsilon_0 = L(s)/F(\tau s)\varphi_0 \) (see Fig. 1), then one has

\[
\varepsilon_0 = \varphi_0^{\alpha} M L(s) [u - u^*] + \beta + \varepsilon_0^P, \quad \beta := \varphi_0^{\alpha} M L(s) [1 - F(\tau s)] F^{-1}(\tau s) * (u - u^*), \quad (11)
\]

and \( \varepsilon_0^P \) is an exponentially decaying term due to the initial conditions \( X_e(0) \) and \( x_f(0) \), which can be bounded by

\[
|\varepsilon_0^P| \leq R_e e^{-\lambda c (t-t_e(\tau))}, \quad (13)
\]

where \( R_e = k_a(|X_e(0)| + |x_f(0)|) \) is independent of \( \tau \), \( 0 < \lambda_c < \min\{\lambda_c, 1/\tilde{\tau}\} \) and \( 0 < \lambda_c < \min\{\lambda_l[A_e]\} \). The inequality (13) holds \( \forall t \in [t_e, t_M) \) where \( t_e(\tau) \in K \) is an upper bound for the peak extinction time \( t_e \) obtained from the known upper bounds of the plant parameters [4].

**VII. MONITORING FUNCTIONS**

The monitoring function is based on a norm bound for \( \varepsilon_0 \) obtained from [3, Lemma 1], valid \( \forall t \in [t_e, t_M) \), if \( f \geq |u^*| \) (modulo vanishing terms) and \( \text{sgn}(k_p) = \text{correct} \).

**A. Norm Bound for \( \varepsilon_0 \)**

Let \( \gamma(t) \) be an absolutely continuous function satisfying

\[
\gamma(t) \geq |\beta(t)| + |\varepsilon_0^P(t)| \quad \text{and} \quad \frac{d}{dt} \gamma(t) \geq -a_m \gamma(t), \quad (14)
\]

\( \forall t \in [t_e, t_M) \). Applying [3, Lemma 1] to (11) and considering \( ML(s) = k_m/(s + a_m) \) with \( a_m, k_m > 0 \), then \( \forall t, t_k \) such that \( t_k \leq t_k \leq t < t_M \), one has \( |\varepsilon_0(t)| \leq \xi(t) \), where

\[
\xi(t) := (|\varepsilon_0(t_k)| + |\beta(t_k)|)e^{-a_m(t_t_k)} + 2\gamma(t).
\]

(15)

the control sign.

perturbed by a uniformly bounded output measurement disturbance \( \bar{\beta} \) for relative degree compensation with a monitoring scheme (\( \varphi_m \)) to adjust the control sign.
VIII. STABILITY RESULT

Considering (4) and (11), the following state vector $z$ is defined $z^T := [X^T, x_f, (z^0)^T]$, where $z^0$ denotes the transient state [13] corresponding to the filters used in (10). Using only linear lead filter compensation, the main stability and convergence result is now stated.

**Theorem 1:** Assume that (A1)–(A4) hold, $f(t)$ is given in (10) and $ML(s) = k_m/(s + a_m)$ with $k_m, a_m > 0$. Then, for sufficiently small $\tau > 0$, the switchings of the control sign, driven by the monitoring function (21), stop after a finite number of switchings. The complete error system (4), (9) and (11), with state $z$, is semi-globally asymptotically stable with respect to a compact set and $z(t)$ is ultimately exponentially convergent to a small residual set of order $O(\tau)$, both sets being independent of the initial conditions. If the nonlinearity in the system satisfies a global Lipschitz condition, the stability properties become global.

**Proof:** See the Appendix.

A. Chattering Avoidance and Exact Tracking

In Fig. 1, the block $L_{red}$ represents the “exact lead filter” which implements $L(s)$ by using RED. Then, ideal sliding mode is reached. The RED for $\dot{e}_0$ and $\ddot{e}_0$ is given by:

\[
\begin{align*}
\dot{\eta}_0 &= v_0, \\
v_0 &= -\lambda_0 |\eta_0 - e_0|^\frac{\tau}{2} \text{sgn}(\eta_0 - e_0) + \eta_1, \\
\dot{\eta}_1 &= v_1, \\
v_1 &= -\lambda_1 |\eta_1 - v_0|^\frac{\tau}{2} \text{sgn}(\eta_1 - v_0) + \eta_2, \\
\dot{\eta}_2 &= -\lambda_2 \text{sgn}(\eta_2 - v_1),
\end{align*}
\]

where $\eta_0(t) \to \dot{e}_0(t), \eta_1(t) \to \dot{e}_0(t) + \eta_2(t) \to \ddot{e}_0(t)$. Higher order differentiators can be found in [6]. We can now state the stability and exact tracking result with the hybrid lead filter.

**Corollary 1:** With the GRED, $\ddot{\beta}(t) = \ddot{\beta}(t) + \beta_\alpha$, and $\varphi_m(t) := (|\ddot{e}_0(t_k)| + |\ddot{\beta}(t_k)|)e^{-a_m(t-t_k)} + a(k)e^{-\lambda(t-t_k)} + 2f_d(t)$, defined with respect to $\dot{e}_0$ (see Fig. 1), all results of the **Theorem 1** hold. Moreover, for an appropriate modulation function, exact tracking is achieved at least exponentially and the control sign switching stops at the correct sign.

**Proof:** See Appendix.

IX. MULTIVARIABLE SYSTEMS

This section presents a first generalization of the proposed controller for MIMO plants with relative degree one and unknown HFG matrix $K_p$. In the MIMO case, the tracking error equation (4) is given by

\[
e_0 = W_M(s)K_p[u - a^*],
\]

where $W_M(s) = \text{diag}\{1/(s + a_i)\}$, with $a_i > 0$ ($\forall i = 1, ..., m$), is the reference model and $e_0, u^* \in \mathbb{R}^m$ are defined as in (4).

According to [13, Lemma 1], if $-K_p$ is Hurwitz, by using the regular form [8] for (25), one can concluded that

\[
|e_0(t)| \leq |e_0(t_k)|e^{-a_m(t-t_k)} + \pi,
\]

where $\pi$ is an exponentially decaying term due to the system initial conditions, $0 < a_m < \min\{|a_i|\}$ and $t_k \in [0, t_M]$ is an arbitrary initial time.

For unknown $K_p$, a collection of matrices $S_q, q \in \mathbb{Q}$ is defined, where $\mathbb{Q}$ is a finite index set such that $-K_pS_q$ is Hurwitz for some $q \in \mathbb{Q}$. Based on (26), the monitoring function in [1] can be extended to MIMO systems, as follows

\[
\varphi_M(t) = |e_0(t_k)|e^{-a_m(t-t_k)} + (k+1)e^{-t/(k+1)}.
\]

Since the relative degree one case is peaking free, the switching time $t_k$ is defined by (22), without the interval $[0, t_0]$, replacing $e_0$ by $e_0$. The switching time $t_k$ sets the change of index $q \in \mathbb{Q}$, thus cycling through the $S_q$ matrices [14]. We adopt the unit vector control (UVC) law [13]

\[
u = -S_qg(e_0, t) e_0 |e_0|^{-1},
\]

where the modulation function $g$ satisfies, $\forall t \in [0, t_M)$,

\[
g(e_0, t) \geq \delta + c_e|e_0(t)| + (1 + c_d)|S_q^{-1}u^*(t)|,
\]

and an upper bound for $|u^*|$ can be obtained applying [12, Lemma 2] to (5), with $W_1 \equiv 0$, and using (7).

With the usual assumptions of multivariable MRAC [9], assume that (A3) holds for the MIMO case, as in [10, pp. 199] and let $x_E$ be the state of any stabilizable and detectable realization of the error system (25).

**Theorem 2:** Consider the UVC law (28) and the monitoring function (27). If the modulation function satisfies (29), then the control sign switching stops and the error system (25), with state $x_E$, is globally asymptotically stable with respect to a compact set and ultimately exponentially convergent to zero. Moreover, if $\delta > 0$, then the sliding mode at the manifold $e_0 = 0$ is reached in some finite time $t_\delta \geq 0$.

**Proof:** See Appendix.

**Remark 1:** We know that if $-K_pS_q$ is Hurwitz all trajectories of the system converge to the origin of the error state space [13, Lemma 1]. Moreover, if $-K_pS_q$ is not Hurwitz, then for almost every initial condition (i.e., except for a set of zero measure) the error trajectories diverge unboundedly or do not converge to the origin. This is a contradiction, since if the switching stops, according to Theorem 2, the state must converge to the origin. Then, almost always, the ultimate $-K_pS_q$ matrix selected at $k = k^*$ must be Hurwitz.
X. Simulation Results

The following examples illustrate the performance of the proposed controllers.

Example 1: (SISO) Let \((A_p, B_p, C_p)\) be a canonical controllable realization of \(G(s) = 1/(s + 2)(s + 1)(s - 1)\) and \(\phi^T = [x_{p1}^T \ 0 \ x_{p2}^T \sin(2t)]\). Here, in contrast to [4], the disturbances can even depend on unmeasurable state \(x_{p2} = \dot{y}\).

The modulation function is implemented using (10), (8) and (6), following the same steps presented in [10]. In particular, to compute the FOAF’s parameters, one can use a simple technique based on Lyapunov quadratic forms, or a less conservative technique based on optimization methods [12].

The switching law \(\alpha(\tilde{e})\), with \(\tilde{e} := \bar{e}_0 - \bar{e}_0\), is taken without boundary layer [5]. In order to counteract the output disturbance introduced by the GRED, the constant \(\beta_0\) is added to the term \(\bar{\beta}\). The other controller parameters are summarized in Table I.

<table>
<thead>
<tr>
<th>Element</th>
<th>Value</th>
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<tbody>
<tr>
<td>Reference model</td>
<td>(M(s) = \frac{4}{(s + 2)(s + 1)(s - 1)}), (r(t) = 5\sin(t))</td>
</tr>
<tr>
<td>Monitor (21)</td>
<td>(a(k) = k + 1), (a_m = 2), (\lambda_c = 1), (t_1 = \ell_c = 0.1s)</td>
</tr>
<tr>
<td>Lead filter</td>
<td>(L(s) = (s + 2)^2), (F(r,s) = (s^2 + 1)^2), (\tau = 10^{-3})</td>
</tr>
<tr>
<td>RED parameters</td>
<td>(\lambda_0 = 3C_1^2), (\lambda_1 = 1.5C_2^2), (\lambda_2 = 1.1C_3), (C_3 = 250)</td>
</tr>
<tr>
<td>GRED</td>
<td>(\alpha = 0) if (</td>
</tr>
</tbody>
</table>

We initialize with \(y(0) = 10\), \(\dot{y}(0) = 10\), \(\ddot{y}(0) = 10\) and a wrong control direction estimate. Fig. 3 illustrates the advantage of the new monitoring function and the remarkable transient response of the proposed scheme under a change of control direction “on-the-fly”. When \(\varphi_m\) uses \(f_d(t) = 0.8 e^{-\sigma t}\), the transient response of \(e_0\) stays almost unaltered after changes in the control direction at \(t = 2, 4, 6\) and \(8s\), see Fig. 3 (a) (−) and Fig. 3 (b). In contrast, using the conservative \(||\bar{\beta}||\), a significant degradation of the transient in \(e_0\) results, see Fig. 3 (a) (−−) and Fig. 3 (c).

Example 2: (MIMO) Consider a visual servoing system \((n^* = 1)\) for a robot manipulator [15]. Neglecting the robot dynamics (kinematic robot), the end-effector (target) motion in the camera image coordinate system is modelled by:

\[
g(t) = K_p[u + \phi], \quad K_p = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \begin{bmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{bmatrix}
\]

where \(y(t) \in \mathbb{R}^2\) denotes the end-effector position vector in the camera space, \(\psi\) represents the rotation angle of the camera framework with respect to the task-space framework and \(h_i > 0\) \((i = 1, 2)\) are uncertain scaling factors, belonging to a known compact set. Note that \(K_p\) is nonsingular. We have included an input disturbance \(\phi(y) = y^2\) satisfying (A3) (with \(k_2 = 0\)), in order to illustrate the disturbance rejection property of the proposed scheme.

The desired dynamics and target position trajectory in the camera space is defined by the reference model:

\[
y_m(t) = -y_m(t) + \tau, \quad y_m, \tau \in \mathbb{R}^2.
\]

The error dynamics \(e_0 = y - y_m\) is given by

\[
e_0(t) = -e_0(t) + K_p(u - u^*),
\]

where \(u^* = -\phi(y) - K_p^{-1}(y - \tau)\). Note that a bound for \(|u^*|\) can be obtained from a known bound for \(K_p^{-1}\). The monitoring function \(\varphi_M\) (27) has \(a_m = 0.9\). The finite set of matrices \(S_q, q \in Q = \{0, 1, 2, 3\}\) is chosen as:

\[
S_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

For any \(\psi\), \(-K_pS_q\) is Hurwitz for some \(S_q\). So, the usual restriction \(|\psi| < 90^0\) [15] can be removed. Here, the plant initial conditions and reference signals are \(y_1(0) = y_2(0) = 0, \ r_1(t) = 2\cos(t)\) and \(r_2(t) = 2\sin(t)\), \(h_1 = h_2 = 1\) and \(\psi = 90^0\). The modulation function which satisfies (29) was calculated with \(\delta = 0.001\) and \(c_d = 5\), where the upper bound \(\varphi(\psi) = 1.1y^2\) was used for the disturbance \(\varphi\). Note that, according to [13, Corollary 1], if the reference model poles are equal then one can choose \(c_d = 0\).

The target trajectory is illustrated in Fig. 4, where one notices that the tracking is achieved even for \(\psi = 90^0\). The monitoring function and \(|e_0|\) are shown in Fig. 5. Note that, at the third switching \((k = k^* = 3)\), the correct \(S_3\) matrix is selected (\(-K_pS_3\) is Hurwitz) and thereafter \(|e_0| \to 0\).

XI. Conclusions

An output-feedback model-reference sliding mode controller was developed for nonlinear uncertain systems with unknown high frequency gain sign and arbitrary relative degree, generalizing the controller in [1]. A new monitoring function with forgetting is designed and the resulting controller leads to global or semi-global asymptotic stability with respect to some compact set and ultimate exponential convergence of the tracking error to zero with improved transient
response. A generalization for MIMO systems with relative degree one was also briefly described and successfully tested with a simulation example inspired from a robotics visual servoing problem.

XII. APPENDIX

Please refer to [http://www.coep.ufrj.br/~luu/ACC07] for the detailed proofs. In what follows, $k_i > 0$ are constants depending only on the plant-controller parameters and $\Psi_{\cdot}(\cdot) \in \mathcal{K}\infty$.

A. Proof of Theorem 1

Let $\mathbf{x}_m := [y \ g \ \cdots \ y^{(n^*-1)}]$, $\mathbf{x}_p := [y_m \ g_m \ \cdots \ y_m^{(n^*-1)}]$ and $\mathbf{x}_e := x - x_{\text{ref}}$. Consider the state $\mathbf{x}_e$ of (4) defined in Section III below (A4). From the Kalman canonical decomposition of $(A_e, k_1, k_2)$ one has $|x_e| \leq k_1|x_e|$. Therefore, since $x_{\text{ref}}$ is uniformly bounded, then $\mathbf{x}_e$ can be norm bounded affinely in $|X_e|$. From (A3) and the global existence of a diffeomorphism of (A1), one can conclude that $|x_e| \leq \Psi(t, |X_e|) + k_1$, consequently, $|x_e| \leq \Psi(t, |X_e|) + k_2$. Now, from (A3) and $X_e$ given below (A4), one can verify that $|X_e| \leq \Psi(t, |X_e|) + k_1$. Noting that $|\mathbf{x}_e| \leq k_1|X_e| + k_1$, one has $|\mathbf{x}_e| \leq \Psi(t, |X_e|) + k_1$, with $f(t)$ in (10). From the small norm property of $\Psi(t, |X_e|) + k_1$, one can conclude that $|\mathbf{x}_e| \leq \tau \Psi(t, |X_e|) + \mathcal{O}(\tau)$, since $|X_e| \leq |\mathbf{x}_e|$. By continuity, given any $R > 0$, $\exists \tau^* \in [0, \tau_0]$ such that $|\mathbf{x}_e| \leq R$, $\forall \tau \in [0, \tau^*]$. Hence, $\forall \tau \in [0, \tau^*]$, $\Psi(t, x) \leq k_1^R \tau^*$, with $k_1^R > 0$ constant and possibly depending on $R$. Following the proof of [3, Proposition 1], [3, Theorem 1] the switching stops after some finite $k^*$ (related to $[t(0)]$) and $[z(t)] \leq \Psi(t, [z(t)]) + \tau$, for some $\tau > 0$ and $\tau$ (sufficiently small) depending on $k_1^R$. Then, for $R > 0$, $\exists \tau_0 > 0$ such that $|\mathbf{x}_e| \leq \tau_0 \Rightarrow |z(t)| \leq R$, $\forall \tau \in [0, \tau^*]$. Thus, $t_{\text{sw}} = +\infty$ and semi-global stability results. Moreover, by using the small-gain theorem [16], $|z(t)| \rightarrow \mathcal{O}(\tau)$ exponentially, as $t \rightarrow +\infty$.

B. Proof of Corollary 1

The exact differentiator will eventually take over providing the exact estimate of $e_0 := Le_0 + \rho^* M L(s)[u - u^*]$, i.e. $e_0 = e_0$, since the error state ultimately enters the residual set. After that, the system becomes exactly a relative degree one case. Finally, suppose we end up with an incorrect control direction estimate. Then, the equation for $e_0$ can be written as $e_0 = \rho^* M L(s)[u - u^*] + \pi$, where $u^* = \theta^* \omega$. In this case, due to the modulation function $10^{-1}$ ($f > u^* + \delta$), there exists $t_4 < +\infty$ such that $|\mathbf{x}_e| \leq \delta$ and, consequently, $e_0|e_0| > 0$, $\forall t \geq t_4$. Hence $e_0$ diverges as $t \rightarrow +\infty$ for all initial conditions, i.e. $e_0$ would not remain in the residual set. Thus, the control direction is correctly estimated after the last switching $k^*$.

C. Proof of Theorem 2

First, one proves that the switching stops after a finite number of switchings. Indeed, for some finite $k^*$ the term $(k^* + 1)e(t)$ of (27) will upper bound $|\mathbf{x}_e(t)|$ in (26) implying that $|e_0(t)| < \phi(t), \forall t \geq t_4$ and, therefore, the controller will converge to zero, at least exponentially, since $\phi(t)$ converges to zero exponentially. Now, write (25) in the regular form, with state $x^* := [x_0^* \ e_0^*]$ where $x_1^* = \frac{\dot{X}_1}{1 + A_2 e_0^*} + A_1 x_0^*$, $A_2$ is Hurwitz. Thus, $x_1$ is input-to-state stable (ISS) w.r.t. $e_0^*$ and the existence of $e_0^*$ implies $|x_1(t)| < \Sigma_2 |e_0(t)| \leq \Sigma_3 |e_0(t)|$. Also from [13, Proposition 1], one can further conclude that $e_0^*$ becomes identically zero after a finite time $t_4$, provided that $\delta > 0$ in (29). Similar to the proof of [3, Theorem 1], $k^*$ can be related to $|x_0^*(0)| + |x_0^*(t)| < \Sigma_2 |x_0^*(0)| + \epsilon$, for some $\epsilon > 0$. Thus, global stability with respect to the ball of radius $\epsilon$ is guaranteed.

REFERENCES


