PEAKING FREE HIGH GAIN OBSERVER BASED SLIDING MODE CONTROL FOR UNCERTAIN SYSTEMS WITH STRONG NONLINEARITIES

Tiago Roux Oliveira*, Liu Hsu*, Alessandro Jacoud Peixoto*

*Programa de Eng. Elétrica, COPPE/Universidade Federal do Rio de Janeiro

Emails: tiagoroux@coep.ufrj.br, liu@coep.ufrj.br, jacoud@coep.ufrj.br

Abstract— A peaking free output-feedback model reference sliding mode controller is introduced for a general class of uncertain nonlinear systems based on high gain observers and *dwell-time* control activation. Semi-global practical exponential stability and significantly improved tracking transient behavior is obtained. Moreover, a monitoring scheme is proposed to deal with induced peaks generated by *output exogenous disturbances*.

Keywords— uncertain nonlinear systems, output-feedback, sliding mode control, peaking phenomenon.

Resumo— Um controlador baseado em modos deslizantes, modelo de referência e realimentação de saída é introduzido para uma ampla classe de sistemas não-lineares incertos utilizando-se observadores de alto ganho e o conceito de *tempo de parada* na ativação do sinal de controle. Estabilidade exponencial semi-global e bom comportamento transitório são obtidos. Além disso, um esquema de monitoração é proposto com o objetivo de evitar os danosos efeitos de picos gerados por perturbações externas.

Palavras-chave— sistemas não-lineares incertos, realimentação de saída, controle por modos deslizantes, fenômeno de pico.

1 Introduction

The output-feedback tracking problem of nonlinear systems with strong nonlinearities has been a challenging problem and it is not surprising that most of the existing output-feedback results impose restrictive assumptions on the nonlinear vector fields, such as particular growth conditions or existence of a global Lipschitz constant (Mazenc et al., 1994).

In general, to solve the problem, some estimate of the plant state, or at least of the state norm, is necessary. In this respect, high-gain observers have been utilized owing to their robustness to plant uncertainties and arbitrarily small estimation error. However, the price to be paid is the generation of peaking which may potentially lead to either bad transient or even instability when the peaking signal is transmitted to the plant (Sussmann & Kokotović, 1991).

Oh & Khalil (1995), Teel & Praly (1995) and Oh & Khalil (1997) proposed globally bounded control (GBC) strategies, which amounts essentially to saturating the control signal, in order to circumvent the deleterious effects of the peaking phenomena. However, the GBC may not guarantee global stability for general classes of nonlinear systems. In order to increase the stability domain, the control saturation level has to be increased. This in turn can result in unacceptable transients since higher peaking signals are transmitted to the plant.

In this sense, we propose an alternative sliding mode control strategy for a class of nonlinear systems which may include strong (e.g., polynomial) nonlinearities, where the norm estimates are also obtained from high gain observers. The control peaking is avoided by introducing a *dwell-time* (Hespanha et al., 2003; De Persis et al., 2002; Freidovich & Khalil, 2007) in the controller activation. Better transient behavior, semi-global practical stability, smaller residual tracking errors and increased stability domains are achieved when compared with the approach proposed in (Oh & Khalil, 1995; Oh & Khalil, 1997).

Another novelty presented here is with respect to the introduction of a *monitoring scheme* which is used as a *peaking detector* of induced peaks generated by *output exogenous disturbances*. Then, the deleterious effect of such peaking phenomena can be automatically circumvented. Simulations illustrate the effectiveness of the proposed controller.

Notation and Definitions: The Euclidean norm of a vector x and the corresponding induced norm of a matrix A are denoted by $|x|$ and $|A|$, respectively. The $\mathcal{L}_{\infty e}$ norm of signal $x(t) \in \mathbb{R}^n$, is defined as $\|x_{t,t_0}\| := \sup_{t_0 \leq \tau \leq t} |x(\tau)|$; for $t_0 = 0$, $\|x_t\|$ is adopted. The symbol "s" represents either the Laplace variable or the differential operator "d/dt", according to the context. The output of a linear system with transfer function $H(s)$ and input u is written $H(s)u$. Classes $\mathcal{K}$, $\mathcal{K}_\infty$ functions are defined as usual (Khalil, 2002, pp. 144). ISS and ISpS mean Input-to-State-Stable (or Stability) and Input-to-State-Practical-Stable (or Stability), respectively (Jiang et al., 1994). Filippov's definition for the solution of discontinuous differential equations (Filippov, 1964) and the concept of *extended equivalent control* (Utkin, 1978; Hsu et al., 2002) are used throughout the paper.

2 Problem Formulation

Consider a SISO nonlinear uncertain plant given by:

$$\dot{x} = Ax + \phi(x,t) + Bu\,, \quad y = Cx\,, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the measured output and $\phi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is a state dependent uncertain nonlinear disturbance, possibly unmatched. No particular growth condition, such as linear growth or existence of a global Lipschitz constant, is imposed on ϕ. Therefore, strong polynomial nonlinearities can be included. Then, finite-time escape is not precluded *a priori* and for each solution of (1) a maximal time interval of definition is $[t_0, t_M)$, where t_0 is the initial time and t_M may be finite or infinite.

2.1 Basic Assumptions

Without loss of generality, $t_0 = 0$ is the initial time. The uncertain triple (A, B, C) is assumed to be in the canonical controllable form. All uncertain parameters belong to some compact set Ω_p such that the necessary uncertainty bounds are available for design. In Ω_p we assume that: (i) ϕ is locally Lipschitz in x $(\forall x)$, piecewise continuous in t $(\forall t)$ and sufficiently smooth; (ii) (A, B, C) represents a linear plant which is minimum-phase, observable, has known order n, relative degree ρ and known high frequency gain (HFG) sign, as usual in Model Reference Adaptive Control (MRAC) (Ioannou & Sun, 1996). Our main additional assumptions are:

(A1) There exists a global diffeomorphism $(\bar{x}, t) = T(x, t)$, $\bar{x}^T := [\eta^T \ \ \xi^T], \eta \in \mathbb{R}^{n-\rho}$, which transforms (1) into the *normal form* (Khalil, 2002), with $\xi = [y \ \dot{y} \ \ldots y^{(\rho-1)}]^T$ and

$$\dot{\eta} = A_0\eta + \phi_0(x, t)\,, \qquad \dot{\xi} = A_r\xi + B_r k_p[u + d_\phi(x, t)]\,,$$

where $y = C_r\xi$, $k_p := CA^{\rho-1}B$ is the *constant* plant HFG and (A_r, B_r, C_r) is in the Brunovsky's controller form. In the η-dynamics: A_0 is Hurwitz and $|\phi_0| \leq \varphi_0(|\xi|, t)$, with φ_0 being a *known* non-negative function, piecewise continuous in t and $\mathcal{K}$ in $|\xi|$.

According to (A1), the nonlinear plant (1) is minimum phase and has strong relative degree ρ (Isidori, 1995). To obtain norm bounds for the matched disturbance $d_\phi(x, t)$, we further assume that:

(A2) There exist known locally Lipschitz functions $\varphi_{T1}, \varphi_{T2} \in \mathcal{K}_\infty$ and constants $k_{T1}, k_{T2} \geq 0$ such that $|\bar{x}| \leq \varphi_{T1}(|x|) + k_{T1}$ and $|x| \leq \varphi_{T2}(|\bar{x}|) + k_{T2}$.

(A3) There exists a known non-negative function $\varphi_d(|x|, t)$ piecewise continuous in t and $\mathcal{K}_\infty$ in $|x|$ such that $|d_\phi(x, t)| \leq \varphi_d(|x|, t)$.

Note that (A2)–(A3) are not restrictive since T, T^{-1} and d_ϕ are continuous in its arguments. Moreover, no particular growth condition is imposed on the bounding functions $\varphi_{T1}, \varphi_{T2}$ and φ_d.

2.2 Control Objective

The aim is, by output-feedback, to achieve semi-global stability properties in the sense of uniform signal boundedness and asymptotic output tracking, i.e., the *output tracking error*

$$e(t) = y(t) - y_m(t) \tag{2}$$

should tend to zero or to some small residual values. The *desired trajectory* $y_m(t)$ is generated by the following *reference model*:

$$y_m = M(s)r = \frac{k_m}{L(s)(s + a_m)}\, r\,, \quad k_m, a_m > 0\,, \tag{3}$$

where $r(t)$ is assumed piecewise continuous, uniformly bounded and the Hurwitz polynomial $L(s)$ is given by

$$L(s) := s^{\rho-1} + a_{\rho-2}s^{\rho-2} + \ldots + a_0\,. \tag{4}$$

2.3 Output Error Equation

Let the minimal realization of $M(s)$ in (3) be given by:

$$\dot{\xi}_m = A_m\xi_m + B_m k_m r\,, \quad y_m = C_m\xi_m\,, \tag{5}$$

where $\xi_m^T := [\, y_m \ \ \dot{y}_m \ \ \ldots \ y_m^{(\rho-1)}\,]$, $B_m := B_r$, $C_m := C_r$ and $A_m := A_r + B_r K_m$, with K_m obtained from the coefficients of the characteristic polynomial of $M(s)$. Now, consider the ξ-dynamics of the plant in (A1). Replacing u by $u + K_m\xi/k_p - K_m\xi/k_p$, we obtain:

$$\dot{\xi} = A_m\xi + B_m k_p[u - K_m\xi/k_p + d_\phi]\,, \quad y = C_m\xi\,. \tag{6}$$

From (5) and (6), the state tracking error $x_e := \xi - \xi_m$ and the output tracking error e satisfy

$$\dot{x}_e = A_m x_e + k_p B_m[u + d]\,, \quad e = C_m x_e\,, \tag{7}$$

$$e = k^* M(s)[u + d]\,, \quad k^* = k_p/k_m\,, \tag{8}$$

where the *equivalent input disturbance* is defined by

$$d(x, t) := -K_m\xi/k_p + d_\phi(x, t) - r/k^*\,. \tag{9}$$

3 Sliding Mode Control

In this section, we describe the state-feedback and the output-feedback sliding mode control approaches, pointing out the *sliding surface* and *modulation function* designs.

3.1 State-Feedback Control

When x and ξ are available for feedback we choose

$$\sigma = Sx_e = 0\,, \quad S := [\ a_0 \ \ \ldots \ \ a_{\rho-2} \ \ 1\ ]\,, \tag{10}$$

as the sliding surface, with $a_0, \ldots, a_{\rho-2}$ defined in (4). From (4), (7) and (10) one can conclude that (A_m, B_m, S) is a non-minimal realization of $k_m^{-1}ML(s)$ and

$$\sigma = k^* ML[u + d]\,. \tag{11}$$

Since $ML(s)$ is strictly positive real, applying (Hsu et al., 1997, Lemma 1) to (11) with control signal $u = -[\mathrm{sgn}(k_p)]\varrho(x, t)\mathrm{sgn}(\sigma(t))$, global exponential stability (GES) and finite time exact tracking are guaranteed, if the *modulation function* $\varrho(x, t)$ (continuous in its arguments) satisfies

$$\varrho(x, t) \geq |d(x, t)| + \delta\,, \tag{12}$$

modulo exponentially decaying terms, where $\delta > 0$ is an arbitrarily small constant.

3.2 Output-Feedback Control

Sliding surface: when only y is available for feedback, the sliding surface is chosen as

$$\hat{\sigma} := S\hat{x}_e = 0\,, \quad \hat{x}_e := \hat{\xi} - \xi_m\,, \tag{13}$$

with $\hat{\xi}$ being an estimate of ξ provided by an HGO due to its robustness to disturbances and parametric uncertainties.

Modulation (or control gain) function: the control law u is redefined by[1]

$$u = -[\mathrm{sgn}(k_p)]\varrho(\chi, t)\mathrm{sgn}(\hat{\sigma}(t))\,, \tag{14}$$

[1] With some abuse of notation, we have kept the same symbol ϱ for the modulation function.

where $\chi(t)$ is a scalar non-negative absolutely continuous function, obtained from available signals, which upper bounds the plant state norm $|x|$, *modulo* exponentially decaying terms. It would be desirable to obtain a *peaking free norm bound* χ such that the inequality (12) could be satisfied by $\varrho(\chi, t)$, i.e.,

$$\varrho(\chi, t) \geq |d(x,t)| + \delta \,. \tag{15}$$

In contrast to the state-feedback case, inequality (15) is not sufficient to achieve global or semi-global tracking due to the disturbances from the inexact HGO estimation. Let the *estimation error* be defined by

$$\tilde{x}_e := x_e - \hat{x}_e = \xi - \hat{\xi} \,. \tag{16}$$

From (10), (11), (13) and (16), one has $\hat{\sigma} = \sigma - S\tilde{x}_e$ and

$$\hat{\sigma} = k^* ML[u + d] - S\tilde{x}_e \,. \tag{17}$$

As shown in (Oliveira et al., 2008), (15) and (17) will imply only an ISS property from $\tilde{x}_e$ to x_e. Furthermore, the estimate $\hat{\xi}$ provided by an HGO has an ISpS property from x_e to $\tilde{x}_e$, with ISpS-gain given by the HGO small parameter μ. Thus, combining the above ISS properties, global or semi-global tracking can be proved through a small-gain analysis.

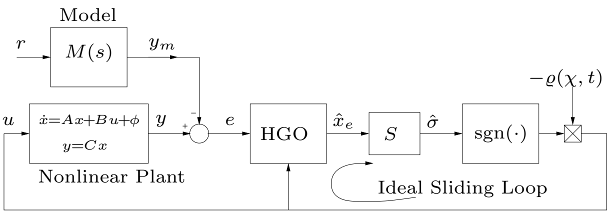

Figure 1: Output-feedback sliding mode controller.

The proposed scheme is depicted in Fig. 1. An eventual peaking (Sussmann & Kokotović, 1991) in $\hat{\sigma}$ is blocked by the sgn(·) function in (14) and the control signal u is peaking free since χ is implemented using only well conditioned (without peaking) available signals.
In the following, we give a detailed description of the proposed controller, stressing the HGO design and the peaking free strategy to obtain χ for classes of nonlinear systems with no particular growth restriction *w.r.t.* the unmeasured states.

4 High Gain Observer

An estimate $\hat{\xi}$ for ξ in (A1) is provided by the HGO:

$$\dot{\hat{\xi}} = A_r\hat{\xi} + k_p^{\text{nom}} B_r u + H_\mu L_o C_r(\xi - \hat{\xi}) \,, \tag{18}$$

where L_o and H_μ are given by

$$L_o := [\; l_1 \quad \ldots \quad l_\rho \;]^T \;\;,\;\; H_\mu := \text{diag}(\mu^{-1}, \ldots, \mu^{-\rho}) \tag{19}$$

and k_p^{nom} is a nominal value for k_p. The observer gain L_o is such that $N(s) = s^\rho + l_1 s^{\rho-1} + \ldots + l_\rho$ is Hurwitz. Since it is desirable that the uncertainties and disturbances have negligible effects in $\hat{x}_e$ (13), the norm of H_μ shall be large, which imposes that μ-constant parameter be small.

4.1 High Gain Observer Error Dynamics

As in (Oh & Khalil, 1997), the following transformation is applied to (16)

$$\zeta := T_\mu \tilde{x}_e \,, \quad T_\mu := [\mu^\rho H_\mu]^{-1} \,, \tag{20}$$

which leads to: (i) $T_\mu(A_r - H_\mu L_o C_r)T_\mu^{-1} = \frac{1}{\mu}A_o$ and (ii) $T_\mu B_r = B_r$, where $A_o := A_r - L_o C_r$. Thus, from the ξ-dynamics in (A1), (16), (18) and (20), one has

$$\mu\dot{\zeta} = A_o\zeta + k_p B_\rho[\mu\nu] \,, \tag{21}$$

with

$$\nu := [\kappa u + d_\phi] \;\; \text{and} \;\; \kappa = (k_p - k_p^{\text{nom}})/k_p \,. \tag{22}$$

4.2 Peaking Phenomenon

As it is well known, HGO estimates may contain peaking (Sussmann & Kokotović, 1991). Indeed, the estimation error $\tilde{x}_e$ will contain a transient term of the form $(a/\mu^b)e^{-ct/\mu}$, for some $a, b, c > 0$. Thus, these terms eventually exhibit an impulsive-like transient behavior, as $\mu \to 0$, where the transient peaks to $\mathcal{O}(1/\mu)$ values before it decays rapidly to zero. This behavior is known as the *peaking phenomenon* (Khalil, 2002; Sussmann & Kokotović, 1991).
However, the peaking phenomenon can be circumvented by using the *peak extinction time* (t_e) concept, where t_e is defined as the solution of $(a/\mu^b)e^{-ct_e/\mu} = 1$, for each value of $\mu \in (0, 1]$. Note that t_e is a function of μ, which satisfies $t_e(\mu) \leq \bar{t}_e(\mu)$, with $\bar{t}_e(\mu) \in \mathcal{K}$ (Cunha et al., 2005). In the next section, this concept will be crucial in the formulation of a peaking free control law.

5 Peaking Free Strategy

In this section, we consider a wider class of nonlinear systems without any type of growth condition imposed on ϕ. As in (Oh & Khalil, 1995; Oh & Khalil, 1997), we will use HGO state not only to define the sliding surface but also use the HGO estimates (with peaking) to design χ and appropriate modulation function $\varrho(\chi, t)$.
In (Oh & Khalil, 1995; Oh & Khalil, 1997), a globally bounded sliding mode control law is applied by using saturation functions in order to avoid the *peaking phenomena*. As a consequence, the region of interest of the control effort must be first estimated in order to tune the saturation level and guarantee semi-global stability. *One major problem in this approach is* that, to enlarge the domain of stability, it is necessary to increase the level of the saturation function. Then, considerable peaking energy is transmitted to the plant, which leads to large transients, stability domain *shrinking* and system degradation performance.

5.1 High Gain Observer plus Dwell-Time

Inspired by the recent developments in *supervisory control* and *logic-based switching* schemes (Hespanha et al., 2003; De Persis et al., 2002; Freidovich & Khalil, 2007), we propose a novel strategy based on the *peak extinction time* and *dwell-time* concepts to cope with the problems induced by peaking, particulary the shrinking of the region of attraction. The new scheme is developed trying to retain the qualities of the state-feedback based sliding mode controller such as good transient performance.

Our **key idea** consists in combining the high gain estimates from HGO with an appropriate dwell-time strategy to obtain a peaking free norm bound χ. In this respect, we only apply the HGO estimates after a certain *dwell-time* τ_D, which is chosen large enough to allow the peaking transients of the HGO to settle down, and small enough to ensure that the trajectories do not leave a prescribed compact set, thus avoiding finite time escape. It will be shown that this choice is possible for μ sufficiently small and

$$\tau_D := \bar{t}_e(\mu)\,, \tag{23}$$

where $\bar{t}_e(\mu)$ is the known upper bound for the peaking extinction time t_e, given in Section 4.2.

5.2 *Norm Bound from HGO*

Due to the high gain properties of the HGO, it can be shown that: while the plant state x remains within any given compact ball, the observer error $\tilde{x}_e$ (16) can be made arbitrary small by reducing the parameter μ. Indeed, the following proposition can be demonstrated.

Proposition 1 *Consider the plant (1) under the assumptions (A1)–(A3) and τ_D defined in (23). Let $t^* \in [0, t_M)$ be the first time such that $|x|$ exits a given ball $\mathcal{B} := \{x : |x| \leq R\}$ of radius $R > |x(0)|$ and $\hat{\xi}$ (18) be the estimate for the state ξ given in (A1). Then, if the HGO parameter μ is sufficiently small such that $\tau_D(\mu) \in [0, t^*)$, one has*

$$|\xi - \hat{\xi}| \leq \tilde{k}^R \mu\,, \quad \tilde{k}^R > 0\,, \quad \forall t \in [\tau_D, t^*)\,, \tag{24}$$

where $\tilde{k}^R$ is a constant possibly depending on R. Moreover,

$$|x(t)| \leq \varphi_{T2}\left(c_0|\hat{\xi}(t)| + \frac{c_1}{s+\lambda_1}\varphi_0(c_2|\hat{\xi}(t)|, t)\right) + \Delta := \chi(t)\,, \tag{25}$$

$\forall t \in [\tau_D, t^*)$, modulo *exponentially decaying terms, where $c_0, c_1, c_2, \lambda_1, \Delta > 0$ are appropriate constants and φ_0, φ_{T2} are given in (A1) and (A2), respectively.*

Proof: The proof of (24) follows from (Oh & Khalil, 1995, Lemma 1) and (25) is a direct consequence of (24), assumptions (A1)–(A2) and the known norm bound for η obtained from application of (Hsu et al., 2003, Lemma 2) to the η-dynamics in (A1). ■

Remark 1 *If the nonlinear system (1) is in the normal form and the η-dynamics in (A1) is absent, the first order filter in (25) can be neglected and the following less conservative upper bound χ can be obtained*

$$\chi(t) := |\hat{\xi}(t)| + \Delta\,, \quad \forall t \in [\tau_D, t^*)\,, \tag{26}$$

where Δ should account only for the effect of the $\mathcal{O}(\mu)$ estimation error in (24).

Thus, from (A2)–(A3), (9) and (25), one can write $|d| \leq \hat{d} + \hat{\pi}$, where $\hat{\pi}$ is a decaying term and

$$\hat{d}(t) := \hat{\varphi}(|\chi(t)|) + c_r\,, \tag{27}$$

with an appropriate constant $c_r > 0$ and $\hat{\varphi} \in \mathcal{K}$. Hence, a peaking free modulation function $\varrho(\chi, t)$ can be obtained:

$$\varrho(\chi, t) = \begin{cases} 0 & , \ \forall t \in [0, \tau_D) \\ \hat{d}(t) + \delta & , \ \text{otherwise}\,, \end{cases} \tag{28}$$

such that (15) holds $\forall t \in [\tau_D, t_M)$.

6 Stability Analysis

In order to fully account for the initial conditions of the *error system* (7) and (21), let:

$$z^T(t) := [z^0(t), x_e^T(t), \zeta^T(t)]\,, \tag{29}$$

$$z^0(t) := [|\eta(0)|, |x_\varrho(0)|]e^{-\gamma t}\,,$$

where z^0 denotes the *transient state* (Hsu et al., 1997) due to state conditions of the stable systems corresponding to the η-dynamics and the filters used in the modulation function design and $\gamma > 0$ is a generic constant. The main stability result is now stated.

Theorem 1 *Consider the error system (7) and (21) with control law (14) and modulation function (28). Assume that (A1)–(A3) hold. Then, for sufficiently small $\mu \in (0, 1]$, the complete error system, with state $z(t)$, is semi-globally exponentially stable* w.r.t. *a small residual set of order $\mathcal{O}(\mu)$ independent of the initial conditions. Moreover, under these conditions, all signals in the closed loop system are uniformly bounded.*

Proof: During the peaking period, the dwell-time control activation protects the plant from peaking, while the state of the plant cannot change by more than $\mathcal{O}(\mu)$ from its initial value. Hence, for sufficiently small μ and $\tau_D(\mu)$ in (23), $x(\tau_D)$ will be arbitrary close to $x(0)$. Then, the proof is based on *Small-Gain Theorem* (Jiang et al., 1994) and follows the same steps in the proof of (Oliveira et al., 2008, Theorem 1). ■

Ideal Sliding Mode: If, additionally to the assumptions of Theorem 1, $\varrho \geq |K_m \xi_m - k_m r| + \delta$ with $\delta > 0$ then the sliding mode $\hat{\sigma}(t) \equiv 0$ is reached in finite time. This implies that finite frequency chattering is avoided.

Absence of Peaking: Since peaking is not transmitted to the plant state x, one can easily conclude that x_e is peaking free by noting that x_e in (7) is ISS with respect to $[u + d(x, t)]$ and that the control law u (14) is peaking free by definition.

Stability Domain and Transient Behavior: In our simulations in Section 8, the stability domain using the proposed approach is larger than one obtained with GBC for the same μ. It is interesting to note that increasing the saturation level, while μ is kept constant, the GBC stability domain is *reduced*. In order to recover the domain using GBC it is necessary to reduce μ, i.e., to increase the HGO gain. However, while the stability domain is increased, the transient behavior is degraded due to large peaks transmitted to the plant allowed by the larger saturation level.

Smaller Residual Set: The residual set in Theorem 1 is of order $\mathcal{O}(\mu)$ while in GBC approach (Oh & Khalil, 1995; Oh & Khalil, 1997) this set is of order $\mathcal{O}(\sqrt{\mu})$.

7 Monitoring Scheme for Peaking Detection

To implement the HGO plus dwell-time strategy, it is necessary to know the arbitrary initial time to keep the control signal equal to zero during the period τ_D. While this is easy when the system is switched on, it may be difficult if the system is subjected to exogenous disturbances that cause an abrupt change in the output of the system and, consequently, induce another peaking on the HGO states during a certain process.

In order to avoid peaking after the initial time, a *monitoring scheme* to detect the onset of peaking is proposed in what follows.

If the system is free of output exogenous disturbances, $|\hat{\xi}| \leq |\xi| + \mathcal{O}(\mu)$ is a natural norm bound for $\hat{\xi}$ obtained from (24), $\forall t \geq \tau_D$. Moreover, as claimed in Theorem 1, ξ tends to ξ_m exponentially. In such case, there exists a finite time $t_a \geq 0$ such that $|\hat{\xi}| \leq |\xi_m| + \mathcal{O}(\mu)$, $\forall t \geq t_a$. A *monitoring function* Φ can thus be defined as

$$|\hat{\xi}(t)| < |\xi_m(t)| + \alpha := \Phi(t)\,, \ \forall t \geq \tau_D\,, \qquad (30)$$

where $\alpha > 0$ is an appropriate design constant such that Φ is a norm bound for $|\hat{\xi}|$, $\forall t \geq \tau_D$. Hence, the *detection time* $\bar{t}_i$ is defined by

$$\bar{t}_{i+1} := \begin{cases} \min\{t > \bar{t}_i : |\hat{\xi}(t)| := \Phi(t)\}, & \text{if it exists}\,, \\ \infty, & \text{otherwise}\,, \end{cases} \qquad (31)$$

where $i \in \{1, 2, \ldots\}$ and $\bar{t}_0 := \tau_D$. The following algorithm in conjunction with (30)-(31) form a *monitoring scheme* to detect and avoid peaking in the control signal valid $\forall t \geq \tau_D$. In a few words, the control signal applied to the plant is equal to zero (just for convenience) during the interval $[\bar{t}_i, \bar{t}_i + \tau_D)$ and is given by (14) and (28) otherwise.

Algorithm for peaking detection, $\forall t \geq \tau_D$

`Step 1.` `Define the initial detection time, say` $\bar{t}_0 := \tau_D$`, the set of indices` $\mathcal{I} := \{0, 1, 2, \ldots\}$ `and the initial value for` $i \in \mathcal{I}$`, say` $i := 0$`.`

`Step 2.` `Put the controller` u `defined by (14) with` $\varrho(\chi, t)$ `in (28) into the loop` $\forall t \geq \bar{t}_0$`.`

`Step 3.` `For` $t \geq \bar{t}_0$`, check continuoulsly the inequality (30). Keep the controller` u `(14) in the loop until` $t = \bar{t}_i (> \bar{t}_0)$ `when the inequality (30) fails according to (31).`

`Step 4.` `Set` $u := 0$`,` $\forall t \in [\bar{t}_i, \bar{t}_i + \tau_D)$`.`

`Step 5.` `Set` $\bar{t}_0 := \bar{t}_i$ `and go back to Step 2.`

We also point also that the proposed monitoring scheme is applicable to a class of exogenous output disturbances, at least stepwise with sufficient time between steps, which avoids multiple peaks during the interval $[\bar{t}_i, \bar{t}_i + \tau_D)$. It would be desirable to characterize more general classes of output disturbances that could be included in the proposed approach.

8 Simulation Results

Consider the second-order nonlinear system in the *normal form* ($\rho = 2$):

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \varepsilon \xi_2^3 + k_p u \\ y &= \xi_1 \end{aligned}$$

In this example, the η-dynamics is absent, the HFG satisfies $|k_p| \leq 2$, $d_\phi = \varepsilon \xi_2^3$ in (A1) and $|\varepsilon| \leq 2$. In the simulations, we consider $\varepsilon = 1$ and $k_p = 0.5$ as the *actual* plant parameters. The reference model is chosen $M(s) = \frac{1}{(s+1)^2}$ with $K_m = [-1 -2]^T$ and $r(t) = 2\sin(\pi t)$. The HGO parameters are: $\mu = 0.001$, $k_p^{nom} = 1$, $N(s) = (s+1)^2$ and $L_o = [2 \ 1]^T$. The equivalent input disturbance d is given in (9). The control gain (28) is computed with δ=1 and

$$\hat{d}(t) = |y| + 2(|\hat{\xi}_2| + \Delta) + 2(|\hat{\xi}_2| + \Delta)^3 + |r|\,, \qquad (32)$$

where $\Delta = 1$ is added to $|\hat{\xi}_2|$ to account for the HGO estimation error of order $\mathcal{O}(\mu)$ (24) after the peaking phase. In this example, the system has a strong nonlinearity in the unmeasured state ξ_2. If the control peaking is transmitted to the plant, finite time escape can be provoked. With the same initial state $x_e(0) = [1 \ \ -1]^T$ as in Fig. 2, the finite time escape occurs at $t = 0.073$ (curves not shown).

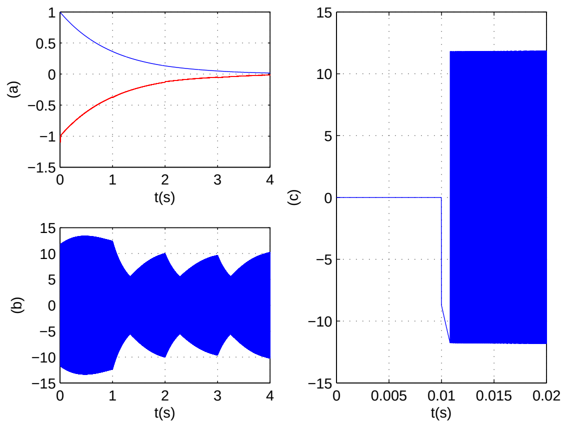

Figure 2: HGO plus dwell-time: (a) state tracking error x_e, (b) control signal u and (c) zoom in u plot showing the dwell-time $\tau_D = 0.01$.

Fig. 2 shows the remarkable performance obtained with the HGO together with the dwell-time strategy ($\tau_D = 10\mu = 0.01$) to generate a peaking free control law. Finite time escape is eliminated for the given initial conditions.

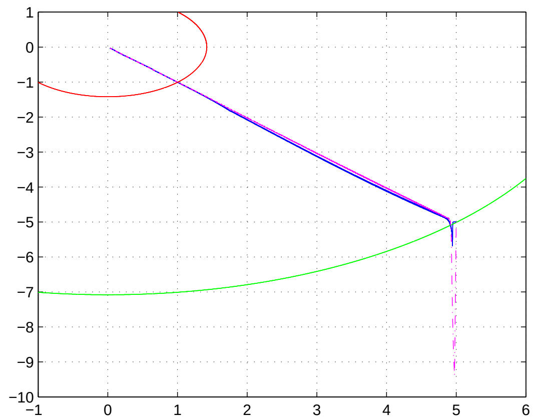

Figure 3: Phase portrait $[x_{e1} \times x_{e2}]$ and stability domain: proposed scheme (solid-line) and GBC (dot-dash) trajectories with initial condition $x_e(0) = [5 -5]^T$.

In Fig. 3, two stability boundaries are shown with $\mu = 0.001$. The larger one (in green line) corresponds to the maximum domain achieved with our proposed scheme. The smaller one (in red line) represents the domain obtained with GBC and $u_{sat} = 500$. In order to achieve the larger stability domain with the GBC, μ must be reduced 3-times. But then, the transient is still degraded. Since the HGO plus dwell-time strategy is not based on saturation, peaking is completely avoided and the stability domain can be arbitrarily increased by reducing μ, without damaging the transient behavior.

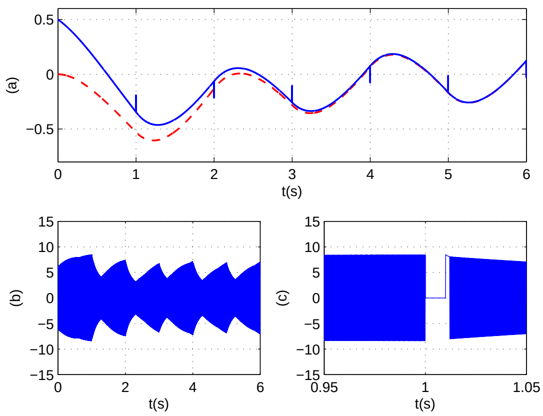

Figure 4: Monitoring scheme and output disturbance: (a) plant output y (solid-line) and reference model output y_m (dash), (b) control signal u and (c) zoom in u plot showing the dwell-time control reinitialization due to the pulse output disturbance at $t=1$.

The HGO plus dwell-time scheme require us to detect any instant where the state suddenly deviates from the current value and program the controller to reinitialize itself starting at that moment. Failing to do so could lead to peaking and the system goes unstable. This situation can be observed when we choose $\xi(0) = [0.5 \; -0.5]^T$ and we insert pulses of amplitude 0.15 and duration 0.0001 to the output y at the time instants $t=1, 2, \ldots, 5$. This disturbance feature was good enough to induce peaking on HGO states and provoke finite time escape at $t = 1.06$ (curves not shown). On the other hand, the dwell-time strategy in conjunction with the *monitoring scheme* (30)-(31) and $\alpha = 2$ handles the situation gracefully and takes care of peaking complications as illustrated in Fig. 4.

9 Conclusions

The sliding mode tracking controller for uncertain nonlinear systems developed in this paper uses HGO estimates in the computation of the switching law and in the control gain design. Due to the dwell-time strategy for control activation, the proposed scheme was shown to be peaking free. The proposed approach was also shown to lead to semi-global exponential stability, *w.r.t.* a small residual set, without the need for globally bounding the control signal. The only parameter required to increase the stability domain is the observer gain. Moreover, a monitoring scheme was proposed to avoid the peaking effects from a class of output exogenous disturbances.

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