Output Feedback Sliding Mode Control for a Class of Uncertain Multivariable Systems with Unmatched Nonlinear Disturbances

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1 Introduction

In this chapter, an output-feedback sliding mode (OFSM) controller for a class of multi-input-multi-output (MIMO) uncertain nonlinear systems is developed. Uncertainties in either the linear or nonlinear terms are allowed. The plant is regarded as a linear system with nonlinear state dependent disturbances. Such disturbances are not necessarily matched with respect to the control inputs.

The classic variable structure control (VSC) design relies on state space representations of the system and the resulting control law implementation requires the availability of the full state vector, e.g., [43]. Owing to the practical difficulty of measuring all states, output-feedback VSC strategies were developed, at first using asymptotic observers [5]. More recently, to cope with uncertain linear and nonlinear systems, sliding mode observers [38, 44, 10] and high-gain observers [11, 12, 35, 31] were proposed.

Alternatively, the unit vector model-reference sliding mode controller (UV-MRAC) follows the model-reference adaptive control (MRAC) approach [37] without the use of state observers [20]. This chapter intends to extend the design of the UV-MRAC, developed for MIMO nonlinear systems of relative degree one in [19], to systems of arbitrary uniform relative degree. The nonlinear unmeasured state dependent disturbances are dealt with according to [33] where one central idea was to reduce all disturbance terms to input disturbances. This leads to a controller design for a class of uncertain MIMO plants with nonlinear state dependent disturbances (locally or globally Lipschitz) which are not uniformly bounded, *a priori.*

The new scheme will be developed trying to retain the advantageous features of the previous UV-MRAC versions. In particular, it is desirable to ensure that no *peaking phenomena* [40] occurs in the closed loop system, as in [19], in contrast to some output feedback controllers based on high gain observers [12, 11].

In addition, since no particular growth restrictions are imposed on the nonlinear terms, the controller should guarantee global (as in [20, 19]) or at least semi-global exponential stability with respect to some small residual set in the error space. The residual set should be arbitrarily small, depending on a design parameter.

The control distribution matrix, and therefore the plant *high frequency gain* (matrix K_p), is allowed to be uncertain. The main motivation to use a unit vector [15, 16] instead of the vector "sign($\cdot$)" switching function, is that a less restrictive prior knowledge of K_p is required, e.g., compared to that obtained using some direct norm-bounds on the uncertainty of the control matrix [10].

This chapter is organized as follows. Section 2 presents some preliminary concepts, definitions, notations and properties. Section 3 describes the plant, the reference model and the control objective. Section 4 briefly introduces the unit vector controller and a key related theorem. A parametrization for the output feedback model matching control and the output error equations are described in Section 5. The development and analysis of the UV-MRAC is carried out in Section 6. Some implementation issues are considered in Section 7 and an illustrative simple simulation example is discussed in Section 8. Conclusions are presented in Section 9 and further reading is suggested in Section 10. The proof of the main result is included in an Appendix.

2 Preliminaries

2.1 Basic concepts and notation

The following basic concepts and notation are employed in this chapter.

- $\|x\|$ denotes the Euclidean norm of a vector x and $\|A\| = \sigma_{max}(A)$ denotes the induced norm of matrix A, with $\sigma_{max}(\cdot)$ denoting the maximum singular value of the argument.
- The $\mathcal{L}_{\infty e}$ *norm* of the signal $x(t) \in \mathbb{R}^n$ is defined as

$$\|x_t\|_\infty := \sup_{0 \leq \tau \leq t} \|x(\tau)\| \ . \tag{1}$$

- Classes of $\mathcal{K}, \mathcal{K}_\infty$ and $\mathcal{KL}$ functions are defined according to [26, p. 144], as follows. A function $\Psi : [0, \sigma_1) \to [0, \infty)$ is of class $\mathcal{K}$ if Ψ is continuous, strictly increasing and $\Psi(0) = 0$. It is of class $\mathcal{K}_\infty$ if additionally it is defined in $[0, \infty)$ and it is unbounded. A function $\mathcal{V} : [0, \sigma_1) \times [t_0, \infty) \to [0, \infty)$ is of class $\mathcal{KL}$ if, for each fixed t, $\mathcal{V}(\sigma, t)$ is of class $\mathcal{K}$ with respect to σ and for each fixed σ, $\mathcal{V}(\sigma, t)$ is monotone decreasing to zero as t increases. Throughout the chapter, the class $\mathcal{K}$ can be assumed to be locally Lipschitz.

- The symbol s denotes either the complex variable in Laplace transforms or the differential operator $\frac{d}{dt}$ in time-domain expressions.
- The *stability margin* λ_0 of a polynomial $p(\lambda)$ is defined as

$$\lambda_0 := \min_i \{-Re(\lambda_i)\} \,, \tag{2}$$

where $\{\lambda_i\}$ are the roots of $p(\lambda)$. Similarly, for a matrix A or a transfer function $G(s)$, a stability margin is defined with $\{\lambda_i\}$ being the eigenvalues of A or the poles of $G(s)$. If $\lambda_0 > 0$, then the polynomial $p(\lambda)$ and the matrix A are said to be Hurwitz and, the transfer function $G(s)$ is bounded-input-bounded-output (BIBO) stable.
- As usual in adaptive control theory, mixed time domain and Laplace transform domain (operator) representations will be adopted. As in [24, 21]: the output y of a linear time invariant system with transfer function $H(s)$ and input u is given by $H(s)u$. Pure convolution operations $h(t) * u(t)$, $h(t)$ being the impulse response from $H(s)$, will be eventually written, for simplicity, as $H(s) * u$. Consider the realization $\dot{x} = Ax + Bu$, $y = Cx + Du$, of $H(s)$. Then,

$$y(t) = H(s)u(t) = h(t) * u(t) + Ce^{At}x(0) \ , \tag{3}$$

where the exponential term is the homogeneous response of the system ($u(t) \equiv 0$). The state x^0 of the homogeneous system $\dot{x}^0 = Ax^0$ is called the transient state. Note that the convolution operator together with the transient state allows a complete input/output description of the linear system, which accounts for the initial conditions.
- The norm of the operator $H(s)$ is defined as

$$\|H(s)\| := \|h(t)\|_1 = \int_0^{+\infty} \|h(\tau)\| d\tau \ . \tag{4}$$

- It is assumed that $t \in [0, \infty)$ so that $\forall t$ means $\forall t \geq 0$, except otherwise stated.

2.2 Class $\mathcal{K}$ Properties

The following properties of class $\mathcal{K}$ functions are useful to extend the results of [20] to the nonlinear case. Similar properties can be found in [39].

Property 1. (*Invariance property of filtered functions of class $\mathcal{KL}$*) Let $\mathit{\Pi}_u$ be a class $\mathcal{KL}$ function and consider the impulse response $h(t)$ of a BIBO stable linear SISO filter. For a fixed $\sigma \geq 0$, let $u(t)$ be a signal norm-bounded by $\mathit{\Pi}_u(\sigma, t)$. Then, there exists a class $\mathcal{KL}$ function $\mathit{\Pi}_y$ such that the filtered signal $y(t) = h(t) * u(t)$ is norm-bounded by $\mathit{\Pi}_y(\sigma, t)$, i.e.,

$$|y(t)| \leq |h(t)| * \mathit{\Pi}_u(\sigma, t) \leq \mathit{\Pi}_y(\sigma, t) \,. \tag{5}$$

Proof. The proof follows applying Lemma 1 (Section 7) to the convolution integral. □

Property 2. (*Separability property for class-$\mathcal{K}$ functions*) Let Ψ be a class-$\mathcal{K}$ function and a, b, α be arbitrary positive constants. Then, the inequality

$$\Psi(a+b) \leq \Psi(a+\alpha a) + \Psi(b+\frac{b}{\alpha}),$$

is verified.

Proof. Since Ψ is an increasing function then $\Psi(a+b) \leq \Psi(b+b/\alpha)$ when $a < b/\alpha$. In addition, $\Psi(a+b) \leq \Psi(a+\alpha a) + \Psi(b+\frac{b}{\alpha})$, since Ψ assumes positive values only. Using the same argument for the $a \geq b/\alpha$ case, the same inequality results thus proving the stated property. □

2.3 Basic MIMO Systems Concepts

Let $\{A, B, C\}$ be a realization of a strictly proper and nonsingular $m \times m$ rational transfer function $G(s) = C(sI - A)^{-1}B$.

- The *observability index* of the pair $\{C, A\}$ ($A \in \mathbb{R}^{n \times n},\ C \in \mathbb{R}^{m \times n}$) is the smallest integer ν, $(1 \leq \nu \leq n)$, such that

$$\mathcal{O}_\nu = \begin{bmatrix} C^T & (CA)^T & \cdots & (CA^{\nu-1})^T \end{bmatrix}^T \tag{6}$$

  has full rank. See [25, pp. 356–357].
- The *relative degree* of a MIMO plant is related with the concept of *interactor matrix* $\xi(s)$ associated with $G(s)$ [45]. In particular, if $G(s)$ has *uniform relative degree* $n^* \geq 1$, see [20], then $CA^{n^*-1}B$ is nonsingular and, in addition, $CA^iB \equiv 0$, $\forall i \in \{0, 1, \ldots, n^*-2\}$, if $n^* \geq 2$.
- The matrix $K_p \in \mathbb{R}^{m \times m}$, finite and nonsingular, is referred to as the *high frequency gain* (HFG) matrix and satisfies

$$K_p = \lim_{s \to \infty} s^{n^*} G(s), \tag{7}$$

  when the *interactor matrix* is diagonal ($\xi(s) = s^{n^*} I$) and $G(s)$ has *uniform relative degree.*

2.4 Discontinuous Differential Equations

- Filippov's definition for the solution of differential equations with discontinuous right-hand sides is assumed [13]. Note that the control signal u is not necessarily a function of t in the usual sense when a sliding mode takes place. In order to avoid clutter, $u(t)$ denotes the locally integrable function which is equivalent to u, in the sense of *extended equivalent control* [43], along any given Filippov solution of the closed-loop system which is absolutely continuous *by definition*. Also, along any such solution, u can be replaced by $u(t)$ in the right-hand side of the governing differential equations. The *extended equivalent control* is defined as an equivalent control which applies for any system motion, not necessarily on a sliding surface [20, Section 2.3].

3 Problem Statement

This chapter considers the model-reference control of a nonlinear MIMO plant

$$\begin{aligned} \dot{x}_p &= A_p x_p + \phi(x_p, t) + B_p u\,, \\ y &= C_p x_p\,, \end{aligned} \tag{8}$$

where $x_p \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^m$ is the output, ϕ is a state dependent nonlinear disturbance which can be decomposed as

$$\phi(x_p, t) := f_x(x_p, t) + f_y(y, t)\,. \tag{9}$$

This decomposition indicates the portion of the disturbance depending only on t and on the *measured* output y.

Matrices A_p, B_p and C_p are uncertain, i.e., only nominal values and some uncertainty bounds are available for design. For $\phi \equiv 0$, the plant (8) is assumed controllable and observable. The linear subsystem has transfer function given by $G(s) = C_p(sI - A_p)^{-1}B_p$.

3.1 Basic Assumptions

As in [20], the following *assumptions* are made:

(A1) $G(s)$ is minimum phase, has full rank and is strictly proper.
(A2) The observability index ν of $G(s)$, or an upper bound of ν, is known.
(A3) The interactor matrix $\xi(s)$ is diagonal and $G(s)$ has known uniform relative degree n^* (i.e., $\xi(s) = s^{n^*} I$), with HFG given by K_p as defined by (7).
(A4) A matrix S_p is known such that $-K_p S_p$ is Hurwitz.

The above assumptions are discussed and motivated in [20] for linear systems. Some additional assumptions must be made on the nonlinearities:

(A5) The nonlinear term $\phi(x_p, t)$ is locally Lipschitz in x_p, $\forall x_p$, and piecewise continuous in t, $\forall t$.
(A6) The nonlinear disturbance ϕ satisfies

$$\|\phi(x_p, t)\| \leq k_x \|x_p\| + \varphi(y, t)\,, \quad \forall (x_p, t)\,,$$

where $k_x \geq 0$ is a scalar and $\varphi : \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^+$ is a known function piecewise continuous in t and continuous in y, satisfying $\varphi(y, t) \leq \Psi_\varphi(\|y\|) + k_\varphi$, where Ψ_φ is of class $\mathcal{K}_\infty$ and k_φ is a positive constant.

The decomposition (9) is such that both nonlinear terms f_x, f_y are composed by matched and unmatched nonlinear functions. Structural restrictions (e.g., triangularity) on f_x and sufficient smoothness of the nonlinear terms f_x and f_y will be later required (see Assumption (A7), Section 5.2) for the case of relative degree higher than one.

As already pointed out in [20], Assumption (A4) represents a considerable reduction in the required prior knowledge concerning the plant HFG matrix. In [41, 42, 6], the more restrictive assumption of positive definiteness of $K_p S_p$ (and also symmetry in some approaches) was needed. In fact, if a unit vector controller is used, one cannot do better since the mentioned Hurwitz condition is necessary and sufficient for the stability of the sliding surface, according to [3, 20] (see Theorem 1 in Section 4).

Assumption (A5) is made in order to allow us to develop a control law u that guarantees local existence and uniqueness (in positive time) of the solution of (8). According to Assumption (A6), no particular growth condition is imposed on φ. Thus, one could have, e.g., $\varphi(y) = \|y\|^2$. Finite-time escape is therefore not precluded, *a priori*.

3.2 Reference Model

The reference model is defined by

$$y_M = W_M(s)\, r\,, \qquad r, y_M \in \mathbb{R}^m\,, \tag{10}$$

$$W_M(s) = \text{diag}\left\{(s+\gamma_1)^{-1}, \dots, (s+\gamma_m)^{-1}\right\} L^{-1}(s)\,, \tag{11}$$

$$L(s) = L_1(s) L_2(s) \cdots L_N(s)\,, \qquad L_i(s) = (s+\alpha_i) I\,, \tag{12}$$

$\gamma_j > 0\,,\ (j = 1, \cdots, m)\,,\ \alpha_i > 0\,,\ (i = 1, \cdots, N)\,,$ and $N = n^* - 1$. The reference signal $r(t)$ is assumed piecewise continuous and uniformly bounded. $W_M(s)$ has the same n^* as $G(s)$ and its HFG is the identity matrix.

3.3 Control Objective

The control objective is to achieve global or semi-global asymptotic stability of the error state with respect to the origin of the error space or to some small residual neighbourhood of the origin. In particular, the tracking error

$$e(t) = y(t) - y_M(t) \tag{13}$$

should asymptotically tend to zero or to small residual values.

4 Unit Vector Control

The unit vector control (UVC) law is given by [15, 16]

$$u = -\varrho(x,t)\frac{v(x)}{\|v(x)\|}\,, \quad \|v\| \neq 0\,, \tag{14}$$

where x is the state vector and $v(x)$ is a vector function of the state of the system. The modulation function $\varrho(x,t) \geq 0$ $(\forall x, t)$ is designed to induce a sliding mode on the manifold $v(x)=0$. Henceforth, $u=0$ is set if $v(x)=0$ only to complete the definition of the control law. However, this does not mean that the equivalent control vanishes on the manifold $v(x)=0$.

The main motivation for the application of UVC in MIMO systems follows from the Theorem below which was proved for UVC systems of arbitrary dimension in [3].

Theorem 1. *Consider the system*

$$\dot{x} = K(x)\frac{x}{\|x\|}\,, \tag{15}$$

where $x \in \mathbb{R}^m$*,* $m \geq 1$*,* $K : \mathbb{R}^m \to \mathbb{R}^{m\times m}$*, and* $\det(K(x)) \neq 0$, $\forall x$*. The origin of the state-space of system (15), with bounded* $K(x)$ *and its derivatives, is stable, asymptotically stable or unstable, if and only if the system* $\dot{z} = K(z)z$ *is stable, asymptotically stable or unstable, respectively.*

In particular, if $K(x)$ is constant, then the origin of the UVC system (15) is globally asymptotically stable if and only if K is Hurwitz. In contrast, no necessary and sufficient conditions are known for VSC systems based on the classic vector "sign(·)" switching function (i.e., for $x \in \mathbb{R}^m$, $\text{sign}(x) = [\text{sign}(x_1), \text{sign}(x_2), ..., \text{sign}(x_m)]^T$) of dimension greater than two [20].

Some lemmas regarding the application of unit vector control within the MRAC framework are presented in [20]. These lemmas generalize their SISO counterparts found in [21] and are instrumental for the controller synthesis and stability analysis. Properties 1 and 2, given in Section 2, are fundamental to extend the results of the mentioned lemmas for the case where exponentially decreasing signals are replaced by the more general class of signals norm-bounded by $\mathcal{KL}$ functions.

5 Model Matching Output Feedback Control

When the plant is perfectly known and free of disturbances ($\phi \equiv 0$), a control law which achieves matching between the closed-loop transfer function and $W_M(s)$ is given by the following parametrization, which appears in the adaptive control literature [37]

$$u^* = \theta^{*T}\omega + \theta_4^{*T} r\,, \tag{16}$$

where the parameter matrix θ^* and the regressor vector $\omega(t)$ are given by

$$\theta^{*T} = [\theta_1^{*T} \ \theta_2^{*T} \ \theta_3^{*T}] \, , \quad \omega = [\omega_1^T \ \omega_2^T \ y^T]^T \, , \tag{17}$$

$$\omega_1 = A(s)\Lambda^{-1}(s)u \, , \quad \omega_2 = A(s)\Lambda^{-1}(s)y \, , \tag{18}$$

$$A(s) = [Is^{\nu-2} \ \ Is^{\nu-3} \ \cdots \ Is \ \ I]^T \, , \tag{19}$$

$$\Lambda(s) = \lambda(s)I \, , \tag{20}$$

$\omega_1, \omega_2 \in \mathbb{R}^{m(\nu-1)}$, $\theta_1^*, \theta_2^* \in \mathbb{R}^{m(\nu-1)\times m}$, $\theta_3^*, \theta_4^* \in \mathbb{R}^{m\times m}$ and $\lambda(s)$ is a monic Hurwitz polynomial of degree $\nu - 1$. The matching conditions require that $\theta_4^{*T} = K_p^{-1}$.

5.1 Output Error Equation

The error equation can be developed following the usual approach for SISO MRAC [17, 24]. Consider the following realization of (18)

$$\dot{\omega}_1 = \Phi\omega_1 + \Gamma u \, , \qquad \dot{\omega}_2 = \Phi\omega_2 + \Gamma y \, , \tag{21}$$

where $\Gamma \in \mathbb{R}^{m(\nu-1)\times m}$ and $\Phi \in \mathbb{R}^{m(\nu-1)\times m(\nu-1)}$ with $\det(sI - \Phi) = \det(\Lambda(s)) = [\lambda(s)]^m$. Let

$$X := [x_p^T \ \omega_1^T \ \omega_2^T]^T \, . \tag{22}$$

The open-loop system composed by the plant (8) and the filters (21) can be written as

$$\begin{aligned} \dot{X} &= A_o X + B_o u + B_\phi \phi \, , \\ y &= C_o X \, . \end{aligned} \tag{23}$$

where

$$A_o = \begin{bmatrix} A_p & 0 & 0 \\ 0 & \Phi & 0 \\ (\Gamma C_p) & 0 & \Phi \end{bmatrix} , \ B_o = \begin{bmatrix} B_p \\ \Gamma \\ 0 \end{bmatrix} , \ B_\phi = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} , \ C_o = \begin{bmatrix} C_p \ 0 \ 0 \end{bmatrix} . \tag{24}$$

Then, the regressor vector is given by

$$\omega = \Omega_1 X \, , \qquad \Omega_1 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ C_p & 0 & 0 \end{bmatrix} . \tag{25}$$

Upon substituting u by u^* given by (16) into (23) and including a disturbance cancellation term $W_\phi(s) * \phi$, the following nonminimal realization of $W_M(s)$ is obtained [19]

$$\begin{aligned} \dot{X}_M &= A_c X_M + B_c K_p \left[\theta_4^{*T} r - W_\phi(s) * \phi\right] + B_\phi \phi \, , \\ y_M &= C_o X_M \, , \end{aligned} \tag{26}$$

where $A_c = A_o + B_o \theta^{*T} \Omega_1$, $B_c = B_o \theta_4^{*T}$ and

$$W_\phi(s) = K_p^{-1} \left[W_M(s)\right]^{-1} C_o \left(sI - A_c\right)^{-1} B_\phi \,. \tag{27}$$

Note that A_c is Hurwitz since the reference model is BIBO stable.

The open-loop system (23) can be rewritten as

$$\begin{aligned} \dot{X} &= A_c X + B_c K_p \left[u - \theta^{*T}\omega\right] + B_\phi \phi \,, \\ y &= C_o X \,. \end{aligned} \tag{28}$$

Now, defining the state error X_e and the output error e by

$$X_e := X - X_M \,, \tag{29}$$

$$e := y - y_M \,, \tag{30}$$

and, subtracting (26) from (28), then X_e and e satisfy

$$\begin{aligned} \dot{X}_e &= A_c X_e + B_c K_p \left[u - \bar{u}\right] \,, \\ e &= C_o X_e \,, \end{aligned} \tag{31}$$

where the model matching control $\bar{u}$ is given by

$$\bar{u} = \theta^{*T}\omega + \theta_4^{*T} r - W_\phi(s) * \phi \,. \tag{32}$$

Since $\{A_c, B_c, C_o\}$ is a realization of $W_M(s)$, the error equation can be rewritten in input-output form as

$$e = W_M(s) K_p \left[u - \bar{u}\right] \,. \tag{33}$$

5.2 Equivalent Nonlinear Input Disturbance

In (33) an upper bound of $\bar{u}$ is necessary to design the control signal u. Considering the plant in (8), if the linear subsystem has uniform relative degree $n^* = 1$, then $W_\phi(s)$ is proper and stable. Thus, an upper bound for the equivalent nonlinear input disturbance d_ϕ, defined by

$$d_\phi := W_\phi(s) * \phi \,, \tag{34}$$

can be directly obtained through the application of Lemma 1 (Section 7) and some upper bound for $\|\phi\|$ [19].

However, for systems with uniform relative degree $n^* > 1$, the transfer function $W_\phi(s)$, defined in (27), can be improper. In this case, time derivatives of the nonlinear disturbance ϕ must be taken into account.

From the definition of the reference model W_M, given in (10), one has

$$K_p^{-1} W_M^{-1}(s) = \mathcal{N}_0 s^{n^*} + \mathcal{N}_1 s^{n^*-1} + \cdots + \mathcal{N}_{n^*} \,,$$

where $\mathcal{N}_i \in \mathbb{R}^{m \times m} (i = 0, 1, \ldots, n^*)$ are functions of $\gamma_i (i = 1, \ldots, m)$ and $\alpha_i (i = 1, \ldots, n^* - 1)$. By using the Markov parameters [25] for representing the transfer function

$$C_o(sI - A_c)^{-1}B_\phi = \frac{C_oB_\phi}{s} + \frac{C_oA_cB_\phi}{s^2} + \frac{C_oA_c^2B_\phi}{s^3} + \dots ,$$

$W_\phi(s)$ (27) can be rewritten as

$$W_\phi(s) = W_N s^N + \cdots + W_1 s + W_0 + \bar{W}_\phi(s) , \tag{35}$$

where

$$\bar{W}_\phi(s) := \sum_{i=0}^{n^*} \mathcal{N}_i C_o A_c^{n^*-i}(sI - A_c)^{-1}B_\phi$$

is strictly proper and BIBO stable and $W_j \in \mathbb{R}^{m\times n}$ are given by

$$W_j = \sum_{i=0}^{N-j} \mathcal{N}_i C_o A_c^{N-j-i} B_\phi = \sum_{i=0}^{N-j} \mathcal{N}_i C_p A_p^{N-j-i} , \quad (0 \le j \le N) . \tag{36}$$

The last equality in (36) comes from the identity $C_oA_c^iB_\phi \equiv C_pA_p^i, (i = 0, 1, \dots, n^*)$, obtained from (24).

From (35), consider the term

$$\big(W_N s^N + \cdots + W_1 s + W_0\big) * \phi , \tag{37}$$

with $\phi = f_x + f_y$ according to (9). In what follows, the notation $f_x^{(j)} := \frac{d^j}{dt^j} f_x(x_p, t)$ is adopted, where the derivative is taken along a solution $x_p(t)$ of (8). An analogous notation is used for $f_y^{(j)}$.

In order to deal with the derivatives of the terms $f_x(x_p, t)$ and $f_y(y, t)$ in (37), the following *assumptions* guarantee sufficient differentiability:

(A7.a) The terms $W_j f_y(y, t)$ $(\forall j \in \{1, \cdots, N\})$ are continuous with respect to y and t and their partial derivatives of order up to N are continuous.

(A7.b) The terms $W_j f_x(x_p, t)$ $(\forall j \in \{1, \cdots, N\})$ are continuous with respect to x_p and t and their partial derivatives of order up to N are continuous.

In addition, the following structural *assumptions* preclude the relative degree from being altered by the nonlinear disturbance. These assumptions guarantee that the time derivative of the control signal u does not appear in the equivalent input disturbance d_ϕ, given in (34).

(A7.c) For $n^* \ge 3$, $(N \ge 2)$,

$$\frac{\partial \left[W_j f_x^{(i)}\right]}{\partial x_p} B_p \equiv 0 , \quad (2 \le j \le N), \quad (0 \le i \le j-2) . \tag{38}$$

(A7.d) For $n^* \ge 2$, $(N \ge 1)$,

$$\left\| \frac{\partial \left[W_j f_x^{(j-1)}\right]}{\partial x_p} B_p \right\| \le k_j , \quad (1 \le j \le N) , \tag{39}$$

where k_j are sufficiently small positive scalars.

In particular, for SISO nonlinear plants, Assumption (A7.c)–(A7.d) are satisfied by nonlinear systems in triangular form.

Remark 1. The following relationship underlies the above assumptions:

$$W_j f_x^{(j)} = \frac{\partial \left[W_j f_x^{(j-1)}\right]}{\partial x_p} \dot{x}_p + \frac{\partial \left[W_j f_x^{(j-1)}\right]}{\partial t} \,,$$

for $n^* \geq 2$ and $j \in \{1, 2, \ldots, N\}$.

Note that the time derivatives $f_y^{(i)}$ do not contribute to modify the relative degree. This can be verified as follows. From (36) and Assumption (A7.c) one has

$$C_p A_p^{N-j} \frac{\partial f_x^{(i)}}{\partial x_p} B_p \equiv 0 \,, \quad (2 \leq j \leq N), (0 \leq i \leq j-2) \,. \tag{40}$$

Thus, after some algebraic manipulations using induction, one can conclude that the time derivatives of the output y of order up to N are independent of the control signal u. Noting that $f_y^{(i)}$ ($\forall i \in \{0, 1, \ldots, p\}$ and p a positive integer) are functions of $y, y^{(1)}, y^{(2)}, \ldots, y^{(p)}$ and t only (independent of u), the time derivatives $f_y^{(i)}$ are also independent of the control signal u, thus $f_y^{(i)}$ does not affect the relative degree.

Now, from Assumptions (A7.a)-(A7.d), one has

$$(W_N s^N + \cdots + W_1 s + W_0) * \phi = \bar{f}_1(x_p, t) + \bar{f}_2(x_p, t) u \,, \tag{41}$$

where $\bar{f}_2$ is uniformly bounded, i.e., $\|\bar{f}_2(x_p, t)\| \leq k_{f2} = k_1 + \ldots + k_N$, for the constants k_j of Assumption (A7.d). Now, consider the following additional *assumption*:

(A7.e) $\|\bar{f}_1(x_p, t)\| \leq \Psi_\phi(\|x_p\|) + k_\phi$, where Ψ_ϕ is of class-$\mathcal{K}_\infty$ and k_ϕ is a positive constant.

Assumption (A7.e) does not impose any particular growth condition on the state dependent nonlinearities and, together with Assumption (A7.c)–(A7.d), leads to

$$\|(W_N s^N + \cdots + W_1 s + W_0) * \phi\| \leq \Psi_\phi(\|x_p\|) + k_\phi + k_{f2}\|u\| \,. \tag{42}$$

From (41) and (35), the equivalent nonlinear input disturbance d_ϕ is given by

$$d_\phi = W_\phi(s) * \phi = \bar{f}_1(x_p, t) + \bar{f}_2(x_p, t) u + \bar{W}_\phi(s) * \phi(x_p, t) \,. \tag{43}$$

Since there exists $\bar{u}$, in (32), that gives the perfect model following control law, it is reasonable to restrict the class of admissible control laws by the following assumption

(A8) The control law satisfies the inequality

$$\|u_t\|_\infty \leq \Psi_X(\|(X_e)_t\|_\infty) + k_{red}\,, \quad \forall t \geq 0\,, \tag{44}$$

where Ψ_X is of class $\mathcal{K}_\infty$ and k_{red} is a positive constant.

Note that Assumption (A8) is consistent with the modulation functions to be determined (Section 7) and it allows one to separate the stability analysis from the modulation function implementation.

The class of nonlinear systems considered here is illustrated through the following example.

Example 1. Consider the SISO system

$$\begin{aligned} \dot{x}_{p1} &= x_{p2} + \phi_1(x_{p1}, x_{p2})\,, \\ \dot{x}_{p2} &= x_{p3} + \phi_2(x_{p1}, x_{p2})\,, \\ \dot{x}_{p3} &= u\,, \\ y &= x_{p1}\,. \end{aligned} \tag{45}$$

In this SISO case the linear subsystem has transfer function given by $G(s) = 1/s^3$ and thus, Assumptions (A1)–(A4) are easily verified. In addition, consider nonlinear disturbances ϕ_1 and ϕ_2 that satisfy the locally Lipschitz condition given in Assumption (A5). Without further information about the nonlinearities, one can simply set $f_x = \begin{bmatrix} \phi_1 \; \phi_2 \; 0 \end{bmatrix}^T$ and $f_y = 0$.

Writing the system in the form (8), (A_p, B_p, C_p) can be taken as the canonical controllable realization of $G(s) = 1/s^3$ with $C_p = \begin{bmatrix} 1 \; 0 \; 0 \end{bmatrix}$.

In (38), considering that $B_p = \begin{bmatrix} 0 \; 0 \; 1 \end{bmatrix}^T$, one has

$$\frac{\partial\,[W_2 f_x]}{\partial x_p} B_p = \frac{\partial\,[W_2 f_x]}{\partial x_{p3}}\,.$$

Since f_x does not depend on x_{p3}, this term is null and Assumption (A7.c) is satisfied. In addition, from (36) one has $W_2 = \mathcal{N}_0 C_p$, where $\mathcal{N}_0 = 1$. Since

$$\frac{\partial\,[W_1 f_x]}{\partial x_p} B_p = \frac{\partial\,[W_1 f_x]}{\partial x_{p3}} = 0\,, \quad \text{and} \quad \frac{\partial \left[W_2 \dot{f}_x\right]}{\partial x_p} B_p = \frac{\partial \dot{\phi}_1}{\partial x_{p3}}\,,$$

Assumption (A7.d) is satisfied if $\left\| \frac{\partial \dot{\phi}_1}{\partial x_{p3}} \right\| < 1$. Noting that

$$\dot{\phi}_1 = \frac{\partial \phi_1}{\partial x_{p1}}(x_{p2} + \phi_1) + \frac{\partial \phi_1}{\partial x_{p2}}(x_{p3} + \phi_2)\,, \quad \text{one has} \quad \frac{\partial \dot{\phi}_1}{\partial x_{p3}} = \frac{\partial \phi_1}{\partial x_{p2}}\,.$$

Thus, Assumption (A7.d) is also equivalent to $\left\| \frac{\partial \phi_1}{\partial x_{p2}} \right\| < 1$. Considering

$$\phi_1 = 0.5 \sin x_{p2} + x_{p1}^2\,, \quad \phi_2 = \frac{x_{p1}}{(1 - \theta x_{p2})^2 + x_{p2}^2}\,,$$

where ϕ_2 is borrowed from [29], Assumptions (A6) and (A7) are satisfied, since

$$\left|\frac{x_{p1}}{(1-\theta x_{p2})^2+x_{p2}^2}\right| \leq (1+\theta^2)|x_{p1}|\,, \quad \text{and} \quad \left\|\frac{\partial\phi_1}{\partial x_{p2}}\right\| = 0.5\cos(x_{p2}) < 1\,.$$

As an example of systems violating some of the above assumptions, consider disturbances with polynomial growth bounds depending on unmeasured states, like in $\phi_1(x_{p1}, x_{p2}) = x_{p1}^2 + x_{p2}^2$. In this case, Assumption (A6) is violated.

6 UV-MRAC Design and Analysis

In simple words, our goal is to design a stable control system that generates some approximation of the model matching control $\bar{u}$. Ideally, if $u \equiv \bar{u}$, then the output error signal $e \to 0$ exponentially, as can be seen from the output error equation (33), since $W_M(s)$ is BIBO stable.

The proposed control law is

$$u = u^{\text{nom}} - S_p U_N\,, \tag{46}$$

$$U_N = \varrho_N \frac{\varepsilon_N}{\|\varepsilon_N\|}\,, \tag{47}$$

$$u^{\text{nom}} = (\theta^{\text{nom}})^T\omega + (\theta_4^{\text{nom}})^T r\,, \tag{48}$$

where $S_p \in \mathbb{R}^{m\times m}$ is a design matrix which satisfies Assumption (A4) and θ^{nom} and θ_4^{nom} are nominal values for θ^* and θ_4^*. The nominal control signal u^{nom} allows the reduction of modulation function amplitudes if the parameter uncertainties $\|\theta^* - \theta^{\text{nom}}\|$ and $\|\theta_4^* - \theta_4^{\text{nom}}\|$ are small.

For systems with uniform relative degree one, $N=0$ and $\varepsilon_N = e$ in (47). For this case, the design of the modulation function ϱ_N and the stability analysis of the closed-loop control system are discussed in detail in [19].

For systems of higher uniform relative degree, the control signal U_N and the auxiliary error ε_N are defined according to the controller scheme given in Figs. 1 and 2 [20]. A key idea for the controller generalization to higher relative degree is the introduction of the prediction error

$$\hat{e} = W_M(s)L(s)K^{\text{nom}}\left(U_0 - L^{-1}(s)U_N\right)\,, \tag{49}$$

where K^{nom} is a nominal value of $K = K_p S_p$ and the operator $L(s)$, as given by (12), is such that $G(s)L(s)$ and $W_M(s)L(s)$ have uniform vector relative degree one. The operator $L(s)$ is noncausal but can be approximated by the unit vector lead filter $\mathcal{L}$ shown in Fig. 2.

The averaging filters $F_i^{-1}(\tau s)$ in Fig. 2 are low-pass filters with matrix transfer function given by $F_i^{-1}(\tau s) = [f_{\text{av}i}(\tau s)I]^{-1}$, with $f_{\text{av}i}(\tau s)$ being Hurwitz polynomials in τs such that the filter has unit DC gain ($f_{\text{av}i}(0) = 1$),

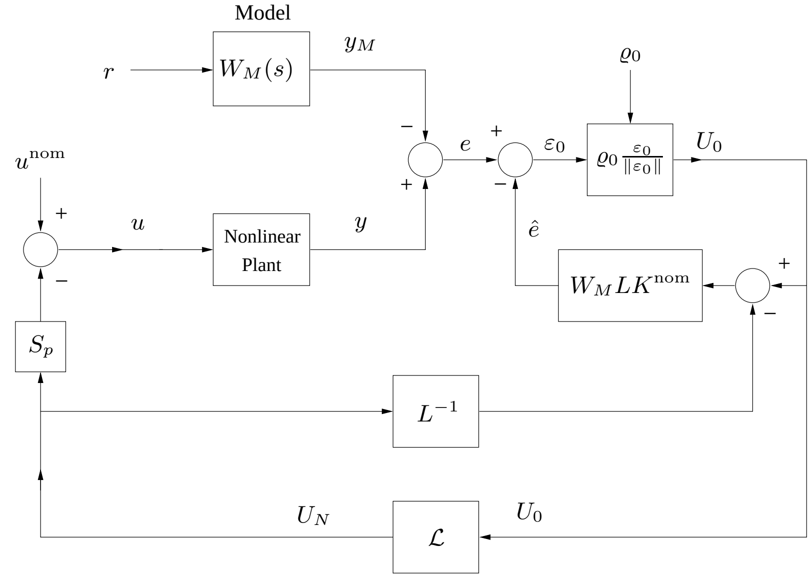

Fig. 1. UV-MRAC for nonlinear plants of uniform relative degree $n^* \geq 2$. The state filters and the computation of $u^{\rm nom}$ and ϱ_0 are omitted to avoid clutter. The realization of the unit vector lead filter $\mathcal{L}$ is presented in Fig. 2.

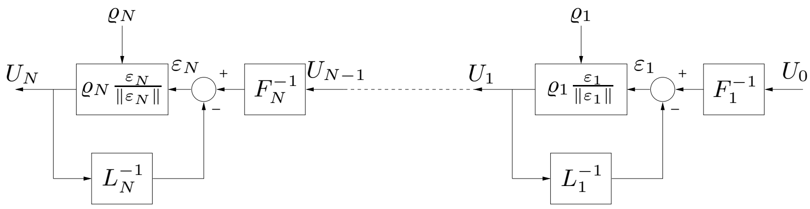

Fig. 2. Realization of the unit vector lead filter $\mathcal{L}$.

e.g., $f_{\text{avi}}(\tau s) = \tau s + 1$. If the time constant $\tau > 0$ is sufficiently small, the averaging filters give an approximation of the equivalent control signals [43]. According to the stability analysis (Section 6.3), the time constant τ is the only design parameter required to increase the region of stability. It is chosen small enough to guarantee the tracking error and the stability region are both acceptable.

6.1 Error Equations

The following expressions for the auxiliary error signals are convenient for the controller design and stability analysis [21]. From (33) and (49), using $u = (\theta^{\rm nom})^T \omega + (\theta_4^{\rm nom})^T r - S_p U_N$, $K = K_p S_p$ and

$$\bar{U} := (K^{\mathrm{nom}})^{-1} K_p \left[(\theta^* - \theta^{\mathrm{nom}})^T \omega + (\theta_4^* - \theta_4^{\mathrm{nom}})^T r - W_\phi(s) * \phi \right] - \\ - \left[I - (K^{\mathrm{nom}})^{-1} K \right] U_N \,, \tag{50}$$

the auxiliary error $\varepsilon_0 = e - \hat{e}$ can be rewritten as

$$\varepsilon_0 = W_M(s) L(s) K^{\mathrm{nom}} \left[-U_0 - L^{-1}(s) \bar{U} \right] \,. \tag{51}$$

The auxiliary errors in the lead filters are given by

$$\varepsilon_i = F_i^{-1}(\tau s) U_{i-1} - L_i^{-1}(s) U_i \,, \quad (i = 1, \ldots, N) \,. \tag{52}$$

These auxiliary errors ε_i $(i = 1, \ldots, N-1)$ and ε_N can be rewritten as:

$$\varepsilon_i = L_i^{-1}(s) \left[-U_i - F_{1,i}^{-1}(\tau s) L_{i+1,N}^{-1}(s) \bar{U} \right] - \pi_{ei} - \pi_{0i} \,, \tag{53}$$

$$\varepsilon_N = -L_N^{-1}(s) (K^{\mathrm{nom}})^{-1} K \left[U_N + F_{1,N}^{-1}(\tau s) U_d \right] - \\ - \left[I - (K^{\mathrm{nom}})^{-1} K \right] \beta_{uN} - \pi_{eN} - \pi_{0N} \,, \tag{54}$$

where $L_{i,j}(s) = \prod_{k=i}^{j} L_k(s)$ $(L_{i,j}(s) = 1$ if $j < i)$, $F_{i,j}(\tau s)$ is defined similarly and (by convention, $\pi_{e1} \equiv 0$)

$$U_d = S_p^{-1} \left[(\theta^* - \theta^{\mathrm{nom}})^T \omega + (\theta_4^* - \theta_4^{\mathrm{nom}})^T r - W_\phi(s) * \phi \right] \,, \tag{55}$$

$$\beta_{uN} = \left[F_{1,N}(\tau s) - I \right] F_{1,N}^{-1}(\tau s) L_N^{-1}(s) U_N \,, \tag{56}$$

$$\pi_{ei} = L_{i-1}(s) F_i^{-1}(\tau s) [\pi_{e,i-1} + \varepsilon_{i-1}] \,, \tag{57}$$

$$\pi_{0i} = \left[W_M(s) F_{1,i}(\tau s) L_{i,N}(s) K^{\mathrm{nom}} \right]^{-1} \varepsilon_0 \,. \tag{58}$$

6.2 Bounds for the Auxiliary Errors

Consider the error system (31), (51), (53), and (54). Let X_ε denote the state vector of (51). The upper bounds for the auxiliary errors are obtained by forcing the modulation functions to satisfy some inequalities (Theorem 2), *modulo* vanishing terms.

Let $x_{\mathcal{F}}^0$ represent these vanishing terms as well as the transient states [21] corresponding to the following operators: L^{-1} in (51), $F_{1,i}^{-1} L_{i+1,N}^{-1}$ in (53), and all the remaining operators associated with $\beta_{uN}, \pi_{ei}, \pi_{0i}$ in (56)–(58). Since all these operators are linear and BIBO stable, there exists a class $\mathcal{KL}$ function $\mathcal{V}_{\mathcal{F}}$ such that

$$\| x_{\mathcal{F}}^0(t) \| \leq \mathcal{V}_{\mathcal{F}}(\| x_{\mathcal{F}}^0(0) \|, t), \quad \forall t \,. \tag{59}$$

In order to fully account for the initial conditions, the following state vector z is used

$$z^T = [(z^0)^T, \varepsilon_N^T, X_e^T] \,, \\ (z^0)^T = [X_\varepsilon^T, \varepsilon_1^T, \varepsilon_2^T, \ldots, \varepsilon_{N-1}^T, (x_{\mathcal{F}}^0)^T] \,. \tag{60}$$

In what follows, "Π" and "Π^0" denote any vanishing terms of the form $\mathcal{V}(\|z(0)\|, t)$ and $\mathcal{V}(\|z^0(0)\|, t)$, respectively, where $\mathcal{V}$ is a generic class $\mathcal{KL}$ function. Since finite escape time cannot be excluded a priori, define $[0, t_M)$ as the maximum time interval of definition of a given solution, where t_M may be finite or infinite.

Henceforth, $\forall t$ means $\forall t \in [0, t_M)$. The following theorem is useful to obtain bounds for the auxiliary errors. For $N = n^* - 1 \geq 1$, assume that $-K^{\text{nom}}$ and $-(K^{\text{nom}})^{-1}K$ are Hurwitz matrices.

Theorem 2. *Consider the auxiliary errors (51), (53) and (54). If the modulation functions satisfy,* $\forall t \in [0, t^*)$, $t^* \leq t_M$,

$$\begin{aligned} \varrho_0(t) &\geq (1 + c_{d0})\|L^{-1}\bar{U}\| + c_{\varepsilon 0}\|\varepsilon_0\|\,, \\ \varrho_i(t) &\geq (1 + c_{di})\|(F_{1,i}^{-1} L_{i+1,N}^{-1})(\bar{U})\|\,, \qquad (i = 1, \cdots, N-1)\,, \\ \varrho_N(t) &\geq (1 + c_{dN})\|F_{1,N}^{-1} U_d\|\,, \end{aligned} \tag{61}$$

modulo *vanishing terms, with some appropriate constants* $c_{\varepsilon 0} \geq 0$ *and* $c_{di} \geq 0$, *for* $i = 0, \ldots, N$, *then,*

$$\|\varepsilon_i(t)\|,\ \|X_\varepsilon(t)\| \leq \Pi^0\,, \tag{62}$$

$$\|\varepsilon_N(t)\| \leq \tau \left\|I - (K^{nom})^{-1}K\right\| k_{eN} C(t) + \Pi\,, \tag{63}$$

and

$$\|\pi_{ei}(t)\|, \|\pi_{0i}(t)\| \leq \Pi^0, \quad (i = 1, \ldots, N)\,, \tag{64}$$

$$\|\beta_{uN}(t)\| \leq \tau k_{\beta N} C(t) + \Pi^0\,, \tag{65}$$

where

$$C(t) := \Psi_X(\|(X_e)_t\|_\infty) + k_{red} \tag{66}$$

with positive constants $k_{eN}, k_{\beta N}$ *and* Ψ_X, k_{red} *as in Assumption (A8). Moreover, if* $t^* \to +\infty$, *the auxiliary errors* ε_i $(i = 0, \ldots, N-1)$ *tend to zero asymptotically.*

Proof. Applying the invariance Property 1, both Corollary 1 and Lemma 2 of [20, p. 292] can be extended to the case in which the exponentially decreasing signals $\pi(t)$ are replaced by the more general class of signals represented by $\mathcal{KL}$ functions. Then the proof follows closely that of Theorem 4 of [20, p. 300] except that, here, t may not be extendable to $+\infty$ if t_M or t^* are finite. □

6.3 Error System Stability

The main result of this chapter is now stated.

Theorem 3. *For $N = n^* - 1 \geq 1$, assume that (A1)–(A8) hold, $-K^{\text{nom}}$ and $-(K^{\text{nom}})^{-1}K$ are Hurwitz matrices, and the modulation functions satisfy (61). Then, for sufficiently small $\tau > 0$, the error system (31), (51), (53) and (54), with state z as defined in (60) is semi-globally exponentially stable with respect to a residual set of order τ, in the sense that there exist $\Psi(\cdot) \in \mathcal{K}_\infty$, a positive function $a(\cdot)$ and a positive constant R_0, which can be chosen arbitrarily large when τ is sufficiently small, such that*

$$\|z(t)\| \leq e^{-a(\|z(0)\|)\,t}\,\Psi(\|z(0)\|) + \mathcal{O}(\tau), \quad \forall t \geq 0\,,$$

provided that $\|z(0)\| \leq R_0$. Under these conditions, all signals in the closed loop system are uniformly bounded.

Proof. See Appendix. □

Now, the global stability result for the UV-MRAC closed loop system can be stated.

Theorem 4. *Under the conditions of Theorem 3 and, in addition, if Assumption (A7) holds with (A7.e) being satisfied by a norm-bound consisting of a globally Lipschitz class-$\mathcal{K}_\infty$ function plus some positive constant, then, for sufficiently small $\tau > 0$, the error system (31), (51), (53) and (54), with state z as defined in (60) is globally exponentially stable with respect to a residual set of order τ, i.e., there exist positive constants a and k_z such that $\|z(t)\| \leq k_z e^{-at}\|z(0)\| + O(\tau)$, $\forall t \geq 0$, and $\forall z(0)$; Under these conditions, all signals in the closed loop system are uniformly bounded.*

Proof. Under the global Lipschitz assumption, R_0 in Theorem 3 can be infinite since in the proof of the former theorem all class-$\mathcal{K}_\infty$ bounds can be linear.□

Remark 2. If the global Lipschitz condition of Theorem 4 is not satisfied, global exponential stability can still be achieved if the plant has uniform relative degree $n^* = 1$, see [19].

Remark 3 (Absence of peaking). From the expression of the state norm-bounds given in Theorems 3 and 4 the transient is independent of τ. This guarantees that the peaking phenomena is absent in the proposed scheme.

6.4 The realization of ideal sliding modes

The realization of ideal sliding modes is important in order to preclude finite frequency chattering in the control signal, at least under ideal conditions. The ideal sliding modes of the UV-MRAC are realized when the auxiliary errors $\varepsilon_i \equiv 0$ ($i = 0, \ldots, N$). From Theorem 2, for $N \geq 1$ the first N sliding modes ($i = 0, \ldots, N-1$) are reached asymptotically. One can further show that the last sliding mode $\varepsilon_N \equiv 0$ is also reached asymptotically according to the following corollary.

Corollary 1. *For $N \geq 1$, under the assumptions of Theorem 3 and with appropriate modulation function ϱ_N, the auxiliary error $\varepsilon_N(t)$ also tends to zero asymptotically if the following transfer function is minimum phase, i.e., has all its transmission zeros with negative real parts:*

$$L_N^{-1}(s)[I + \Delta K F^{-1}(\tau s)]\,, \tag{67}$$

where $\Delta K = (K^{\mathrm{nom}})^{-1}K - I$ and $F(\tau s) = F_{1,N}^{-1}(\tau s)$.

Proof. The equation for ε_N can be rewritten as

$$\varepsilon_N = -L_N^{-1}[I + \Delta K F^{-1}]\{U_N + [I + \Delta K F^{-1}]^{-1}(K^{\mathrm{nom}})^{-1}KF^{-1}U_d\} - \\ - \pi_{eN} - \pi_{0N}\,. \tag{68}$$

The corollary follows from direct application of Lemma 1 of [20, p. 291]. □

A simple sufficient condition for the corollary to hold comes from the application of the small gain theorem to the transfer function $\Delta K F^{-1}(\tau s)$ with unitary feedback. This results in the sufficient condition:

$$\|\Delta K\|\;\|F^{-1}(\tau s)\|_\infty < 1\,, \tag{69}$$

where $\|\cdot\|_\infty$ denotes the $\mathcal{H}_\infty$ norm. The modulation function ϱ_N should satisfy (61) and also, according to Lemma 1 of [20, p. 291],

$$\varrho_N(t) \geq (1 + c_{dN1})\|[I + \Delta K F^{-1}]^{-1}(K^{\mathrm{nom}})^{-1}KF^{-1}U_d\|\,. \tag{70}$$

As will be clear from Lemma 1, a simple first order filter can be designed to calculate ϱ_N from $\|U_d\|$.

7 Modulation Functions

According to the definition of the signals $\bar{U}$ (50) and U_d (55), the modulation functions that satisfy (61), *modulo* vanishing terms, can be implemented using upper bounds for ε_0 and for the output of the filters L^{-1}, $F_{1,i}^{-1}L_{i+1,N}^{-1}$ and $F_{1,N}^{-1}$ driven by the measured signals ω, r, U_N and by an upper bound $\hat{d}_\phi$ of the equivalent input disturbance nonlinear term $W_\phi(s) * \phi$.

Upper bounds for the output signal of stable linear systems (possibly uncertain) can be generated through a *first order approximation filter* (FOAF), as presented in the following lemma that extends the applicability of [23, Lemma 3.1] to multivariable systems [8, 21]. In this lemma the system

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \qquad y(t) = Cx(t) \tag{71}$$

is considered, where $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$. The system transfer function is given by $G(s) = C(sI - A)^{-1}B$ and $g(t)$ is the system impulse response. Let λ_0 be the stability margin of matrix A and $\lambda := \lambda_0 - \delta$ with $\delta > 0$ being an arbitrary constant. Let $\bar{u}(t)$ be an instantaneous upper bound of $u(t)$, i.e., $\|u(t)\| \leq \bar{u}(t)$, $\forall t$. Note that, in the following lemma $G(s)$ is not necessarily stable.

Lemma 1. *For system (71), there exist $c_1, c_2 > 0$ such that the impulse response $g(t)$ satisfies $\|g(t)\| \leq c_1 e^{-\lambda t}$ and the following inequalities hold*

$$\|g(t) * u(t)\| \leq c_1 e^{-\lambda t} * \bar{u}(t)\,, \quad \forall t\,, \tag{72}$$

$$\|y(t)\| \leq c_1 e^{-\lambda t} * \bar{u}(t) + c_2 \|x(0)\| e^{-\lambda t}\,, \quad \forall t\,. \tag{73}$$

Proof. See [19, Appendix IV] and [23]. □

One can further simplify the modulation functions by using Lemma 1 to compute a simple, but more conservative upper bound for $\|\omega\|$, since ω is a filtered version of the signals u and y, see (17)–(20). The FOAF concept is very useful to compute upper bound signals using low order linear filters without the application of complex filtering devices. On the other hand, conservative upper bounds are obtained. Thus, in this approach, there exists a tradeoff between controller complexity and control signal amplitude.

7.1 Norm-Bounds for the Plant State

The synthesis of the UV-MRAC modulation functions require an upper bound $\hat{d}_\phi$ for the equivalent input disturbance $d_\phi := W_\phi(s) * \phi$ term, where ϕ is a function of the unmeasured system state x_p. Thus, an estimate of $\|x_p\|$ must be obtained.

The lemma presented below provides an upper bound for the norm of the state of the nonlinear system (74). In this lemma, the input signal U can be a switched signal generated by a sliding mode control law and the signal U_{av} is the *average control* generated through the auxiliary low-pass filter (75). This lemma considers the system

$$\dot{x} = Ax + B[U + d] + B_\phi \bar{\phi}(x,t)\,, \tag{74}$$

$$\tau_{av}\, \dot{U}_{av} = -U_{av} + U\,, \tag{75}$$

where $U, d \in \mathbb{R}^m$ are input signals, d is locally integrable in the sense of Lebesgue, $\tau_{av} > 0$ is an arbitrary time constant, $x \in \mathbb{R}^n$ and $U_{av} \in \mathbb{R}^m$. The function $\bar{\phi} : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^l$ is piecewise continuous in t, locally Lipschitz in x ($\forall x$) and satisfies

$$\|\bar{\phi}(x,t)\| \leq k_x \|x\| + \bar{\varphi}(t)\,, \tag{76}$$

with $k_x \geq 0$ being some scalar and $\bar{\varphi} : \mathbb{R}^+ \to \mathbb{R}^+$ being piecewise continuous. Let λ_0 be the stability margin of A. Let $c_\phi > 0$ and $\gamma < \lambda_0$ be such that $\|w_\phi(t)\| \leq c_\phi e^{-\gamma t}$, $\forall t$, where $w_\phi(t)$ is the impulse response corresponding to the transfer function $(sI - A)^{-1} B_\phi$.

Lemma 2. *For system (74)–(75), if $k_x < \gamma / c_\phi$, then $\gamma_x := \gamma - c_\phi k_x > 0$, the system (74) is bounded-input-bounded-state stable and $\exists c_i \geq 0$ $(i = 0, \ldots, 6)$ such that the following inequality holds $\forall t$*

$$\begin{aligned}\|x(t)\| \leq\; & c_2 \tau \|U_{av}(t)\| + e^{-\gamma_x t} * [c_3 \bar{\varphi}(t) + (c_4 + \tau c_5)\|U_{av}(t)\| + c_6 \|d(t)\|] \\ & + [c_0 \|x(0)\| + c_1 \tau \|U_{av}(0)\|]\, e^{-\gamma_x t}\,. \end{aligned} \tag{77}$$

Proof. The proof consists in applying Lemma 1 and can be found in [19, Appendix V]. □

Applying Lemma 2 to (28) and considering Assumption (A6), it is possible to find a constant $k_x^* > 0$ such that, for $k_x \in [0, k_x^*]$ a norm-bound for X in (28) can be obtained by using first order approximation filters, resulting in the following inequality

$$\|X(t)\| \leq \hat{X}(t) + \hat{\pi}(t)\,, \tag{78}$$

where the vanishing term $\hat{\pi}$ comes from the initial conditions $X(0)$ in (28) and $U_{av}(0)$ in (80). Note that $\hat{\pi}$ can be bounded by Π^0, a generic class $\mathcal{KL}$ function with $\|z^0(0)\|$ being its first argument.

The norm state estimation $\hat{X}$, *modulo* vanishing term $\hat{\pi}$, is given by

$$\hat{X}(t) := c_2\tau_{av}\|U_{av}\| + \frac{1}{s+\lambda_x}[c_3\varphi + c_\omega\|\omega\| + c_r\|r\| + (c_4 + \tau_{av}c_5)\|U_{av}\|] \tag{79}$$

with appropriate $c_i > 0$ $(i = 2, \cdots, 5)$ defined in Lemma 2, $c_\omega, c_r \geq 0$ are determined from the parameter uncertainties $\|\theta^* - \theta^{\text{nom}}\|$ and $\|\theta_4^* - \theta_4^{\text{nom}}\|$, λ_x being an appropriate positive constant, as detailed in [19], and the averaging control U_{av} is redefined replacing U by U_N in (75), i.e.,

$$U_{av} := \frac{I}{\tau_{av}s + 1}U_N\,. \tag{80}$$

The upper bound (78) applies to the unmeasured norm $\|x_p\|$ recalling that $\|X\|^2 = \|x_p\|^2 + \|\omega_1\|^2 + \|\omega_2\|^2$. Now, an upper bound for $\|\phi\|$ can be derived from $\hat{X}$ through the application of the inequality (78) considering Assumption (A6) [19].

7.2 Upper Bound for the Equivalent Nonlinear Input Disturbance

From Assumptions (A5)–(A7), (42) and (43), the following upper bound is valid

$$\|d_\phi\| \leq \Psi_\phi(\|x_p\|) + k_\phi + k_{f2}\|u\| + \|\bar{W}_\phi(s) * \phi(x_p, t)\|\,. \tag{81}$$

Since $\bar{W}_\phi(s)$ is strictly proper and BIBO stable $\|\bar{W}_\phi(s) * \phi\|$ can be directly bounded using Lemma 1, Assumption (A6) and the upper bound $\hat{X}$, resulting in

$$\|\bar{W}_\phi(s) * \phi(x_p, t)\| \leq c_\phi e^{-\lambda_\phi t} * (k_x\hat{X} + \varphi(y, t))\,, \tag{82}$$

with constants $c_\phi > 0$ and $0 < \lambda_\phi < \lambda_o$, where λ_o is the stability margin of A_c in (31), see Lemma 1, with φ given in Assumption (A6). The term $\Psi_\phi(\|x_p\|)$, in (81), can be bounded by $\Psi_\phi(\|x_p\|) \leq \Psi_\phi(\hat{X} + \Pi^0)$. Through the application of Property 2, the upper bound term $\Psi_\phi(\hat{X} + \Pi^0)$ can be decomposed as a sum of a known term, depending on $\hat{X}$, and some vanishing term, depending on Π^0. Thus, $\|d_\phi\|$ can be bounded by

$$\|d_\phi\| = \|W_\phi(s) * \phi\| \le \hat{d}_\phi + \Psi_\phi((\alpha^{-1}+1)\Pi^0)\,, \tag{83}$$

where

$$\hat{d}_\phi := \Psi_\phi((\alpha+1)\hat{X}) + k_{f2}(\|S_p\|\varrho_N + \|u^{\mathrm{nom}}\|) + \\ + c_\phi e^{-\lambda_\phi t} * (k_x\hat{X} + \varphi(y,t)) + k_\phi\,. \tag{84}$$

7.3 Implementation of the Modulation Functions

Modulation functions that satisfy (61), *modulo* vanishing terms, can be implemented using only the available signals $\hat{X}, \omega$ and r, see [20, Section 7.1]. The additional transient terms due to the filters used to implement the modulation functions can be included in the state z^0, which are considered in the stability analysis.

For the design of $\varrho_i,\ (i = 0,\ldots,N)$, Lemma 1 is applied to the inequalities (61), considering the definition of $\bar{U}$ (50) and U_d (55). For ϱ_0 it gives,

$$\varrho_0 = \delta_0 \ + \ c_{\varepsilon 0}\|\varepsilon_0\| + c_{\omega 0}\|L^{-1}(s)\omega\| + c_{U0}\|L^{-1}(s)U_N\| + \\ + c_{r0}\|L^{-1}(s)r\| + \hat{d}_0(t)\,, \quad \hat{d}_0(t) = \frac{c_{d0}}{s+\lambda_{d0}}\hat{d}_\phi\,, \tag{85}$$

with appropriate constants $c_{\varepsilon 0},\ c_{\omega 0},\ c_{U0},\ c_{r0},\ c_{d0} > 0$, arbitrary $\delta_0 \ge 0$, and λ_{d0} being the stability margin of the filter $L^{-1}(s)$.

Similarly, for $\varrho_i,\ (i = 1,\ldots,N-1)$, one gets

$$\varrho_i = \delta_i \ + \ c_{\omega i}\left\|F_{1,i}^{-1}(\tau s)L_{i+1,N}^{-1}(s)\omega\right\| + c_{ri}\left\|F_{1,i}^{-1}(\tau s)L_{i+1,N}^{-1}(s)r\right\| + \\ + c_{Ui}\left\|F_{1,i}^{-1}(\tau s)L_{i+1,N}^{-1}(s)U_N\right\| + \hat{d}_i(t)\,, \quad \hat{d}_i(t) = \frac{c_{di}}{s+\lambda_{di}}\hat{d}_\phi\,, \tag{86}$$

with arbitrary $\delta_i \ge 0$, appropriate constants $c_{\omega i},\ c_{Ui},\ c_{ri},\ c_{di} > 0$ and λ_{di} being the stability margin of the filter $F_{1,i}^{-1}(\tau s)L_{i+1,N}^{-1}(s)$.

For ϱ_N one has that (61) and (70) must be satisfied. Thus,

$$\varrho_N \ge \delta_N \ + \ c_{\omega N}W_F(\tau s)\,\|\omega\| + c_{rN}W_F(\tau s)\,\|r\| + W_F(\tau s)\hat{d}_\phi\,, \tag{87}$$

with arbitrary $\delta_N \ge 0$, appropriate constants $c_{\omega N},\ c_{rN} > 0$ and $W_F(\tau s)$ being a common FOAF for the transfer functions (applied to U_d) appearing in the last inequality (61) and in (70). Note that the modulation functions $\varrho_i,\ (i = 1,\ldots,N-1)$ are implemented as a function of U_N. Rewriting (84) as

$$\hat{d}_\phi := \bar{d}_\phi + k_{f2}\|S_p\|\varrho_N\,, \tag{88}$$

where $\bar{d}_\phi := \Psi_\phi((\alpha+1)\hat{X}) + k_{f2}\|u^{\mathrm{nom}}\| + c_\phi e^{-\lambda_\phi t} * (k_x\hat{X} + \varphi(y,t)) + k_\phi$, one may argue that (87) has no solution, since it is an implicit inequality in ϱ_N. However, for k_{f2} sufficiently small, the filter $(1 - k_{f2}\|S_p\|W_F(\tau s))^{-1}$ is stable. Thus (87) has a solution given by

$$\varrho_N = (1 - k_{f2}\|S_p\|W_F(\tau s))^{-1}\bar{\varrho}_N\,, \tag{89}$$

where

$$\bar{\varrho}_N = \delta_N \ + \ c_{\omega N}W_F(\tau s)\,\|\omega\| + c_{rN}W_F(\tau s)\,\|r\| + W_F(\tau s)\bar{d}_\phi\,. \tag{90}$$

Remark 4. Note that, from (18) and (20), one can choose a constant matrix $\theta_{av}^T = [I \; n_{\nu-3}I \; \dots \; n_0 I]$, with $\lambda(s) = (s^{\nu-2} + n_{\nu-3}s^{\nu-3} + \dots + n_0)(s+\alpha)$, such that

$$\frac{I}{\tau_{av}s+1}u = \frac{(s+\alpha)I}{(\tau_{av}s+1)}\theta_{av}^T\omega_1\,. \tag{91}$$

From (80) and (46), one has

$$S_pU_{av} = \frac{I}{\tau_{av}s+1}u^{\text{nom}} - \frac{I}{\tau_{av}s+1}u\,,$$

thus, from (91), $\|(U_{av})_t\|_\infty$ can be bounded affinely by $\|\omega_t\|_\infty$.

Now, recalling that the norm-bound $\hat{X}$, defined in (79), for the state vector X holds for k_x sufficiently small, and that (89) is valid for k_{f2} sufficiently small, where k_x is the growth rate described in the condition imposed on the nonlinear term $\phi(x_p,t)$, in Assumption (A6), and $k_{f2} = k_1 + \dots + k_N$, according to Assumption (A7.d). Then, by using the separability Property 2, the restriction imposed on the admissible class of control laws described in (44) is satisfied for k_x and k_{f2} sufficiently small. Thus,

$$\|\varrho_N(t)\| \le \mathit{\Psi}_X(\|(X_e)_t\|_\infty) + k_{red}\,, \tag{92}$$

is in agreement with the Assumption (A8).

8 Simulation Results

Consider the plant (8) with

$$A_p = \begin{bmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0&0\\ b_{21}&b_{22}\\ 0&0\\ b_{41}&b_{42} \end{bmatrix}, \quad C_p = \begin{bmatrix} 1&0&0&0\\ 0&0&1&0 \end{bmatrix},$$

and $\phi(x_p,t)$ given by

$$\phi(x_p,t) = \begin{bmatrix} \eta_1 x_{p1}^2\\ 0\\ \eta_2 x_{p2}\sin(\omega_\phi t)\\ 0 \end{bmatrix},$$

where $b_{41} = b_{42} = 0.1$ and the uncertain parameters are bounded by $0.6 \leq b_{21} \leq 2$, $-6 \leq b_{22} \leq -2$, $1 \leq \eta_1 \leq 3$, $0 \leq \eta_2 \leq 0.5$ and $8\ rad/s \leq \omega_\phi \leq 10\ rad/s$. The nonlinear disturbance f_y is composed of a quadratic function of the output signal $y_1 = x_{p1}$ and is unmatched with respect to the control signal. Therefore, this nonlinear system is locally Lipschitz and the disturbance term ϕ cannot be trivially cancelled by input signals, in view of the uncertainties. The linear subsystem has uniform relative degree $n^* = 2$ and high frequency gain matrix given by

$$K_p = \begin{bmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \end{bmatrix} . \tag{93}$$

It can be verified that the uncertain matrix $-K_p$ is always Hurwitz. Thus, Assumption (A4) is satisfied with $S_p = I$.

The chosen reference model is

$$W_M(s) = \operatorname{diag}\left\{ \frac{1}{(s+4)^2}, \frac{1}{(s+4)^2} \right\} . \tag{94}$$

The nominal control signal is important to allow smaller modulation functions. The nominal parameter matrix was computed in view of the matching of the closed-loop nominal linear subsystem to the reference model, considering $L(s) = \Lambda(s) = (s+10)I$ and assuming that the *nominal* parameter values are $b_{21} = 0.66$, $b_{22} = -2$.

The modulation functions (85)(89)(90) were implemented with $c_{\omega 0} = c_{U0} = c_{r0} = 1$, $c_{\varepsilon 0} = 0$, $c_{\omega 1} = c_{r1} = 0.2$, $\delta_0 = \delta_1 = 0.1$, $k_{f2} = 0$, $K^{\text{nom}} = K_p^{\text{nom}}$. The upper bound $\bar{d}_\phi$ was implemented with $\alpha = 1$, $c_\phi = 0.4$, $\lambda_\phi = 5$, $k_x = 0.5$, $\varphi = 3y_1^2$ and the state *norm-bound* estimate $\hat{X}$, given by (79), with $\tau_{av} = 0.01$, $c_i = 0.5$, $(i = 2, \ldots, 5)$, $c_\omega = c_r = 1$ and $\lambda_x = 5$.

In the simulation results presented in Figure 3, the reference signals r_1 and r_2 are, respectively, a square wave of amplitude 1 and frequency 4 rad/s and a sinusoid of amplitude 1 and frequency 3 rad/s. The *true* plant parameters, assumed unknown, are $b_{21} = 1$, $b_{22} = -3$, $\eta_1 = 2$, $\eta_2 = 0.5$ and $\omega_\phi = 10$. The convergence of the plant output signals to the model reference signals is apparent in Figure 3. The averaging filters time constant $(\tau = 0.01\,s)$ was chosen small enough to keep the region of stability of the closed loop system large and to maintain the residual output error small.

9 Conclusion

In this chapter, an output-feedback model-reference sliding mode controller for a class of uncertain nonlinear MIMO systems was developed. The proposed controller is an extension of the UV-MRAC controller, introduced in [19], to systems of arbitrary uniform relative degree. The central idea of the new UV-MRAC design consists of reducing the nonlinear disturbance terms, possibly

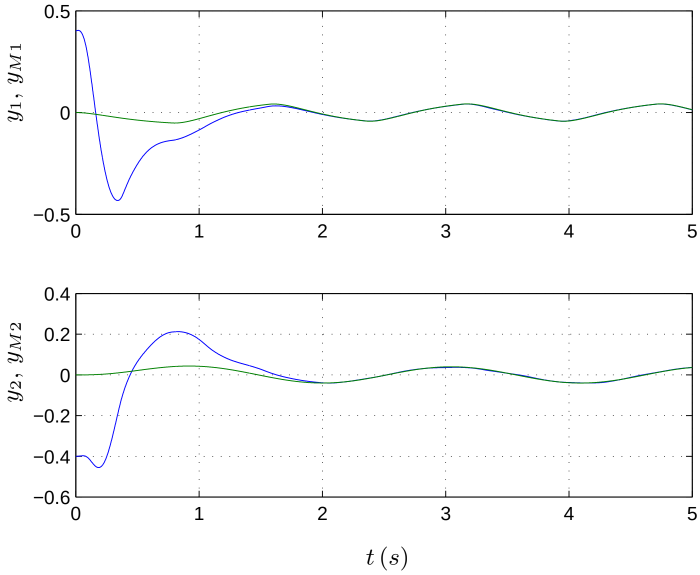

Fig. 3. Simulated plant output signals and reference model outputs.

unmatched, to an equivalent input disturbance. The generalized UV-MRAC was shown to be, in general, semi-globally exponentially stable with respect to a small residual set. The only parameter required to increase the domain of stability is the time constant τ of the averaging filters. This time constant is the analogue of the small parameter that characterizes high gain observers (HGO) applied in output-feedback sliding mode controllers.

In contrast to the controllers based on high-gain observers, the proposed sliding controller is free of peaking in the control signal without resorting to saturation. Moreover, it was shown that the UV-MRAC ensures global exponential stability with respect to a small residual set of order τ when the equivalent nonlinear input disturbance is bounded by some globally Lipschitz class $\mathcal{K}$ function. Simulations illustrate the performance of the proposed scheme in the presence of polynomial output dependent unmatched disturbance. Experimental evaluation remains to be assessed, particularly in the presence of measurement noise. However, from the successful experimental tests already performed with the similar controller VS-MRAC applied to real underwater vehicles [7], it is believed that the UV-MRAC will also perform well, even in the presence of noise in real world applications. One important parameter to attenuate noise effect is the averaging filter time constant, similar to the observer gain of high gain observers.

10 Notes and References

The importance of output-feedback in the application of sliding mode control has been well motivated in [5] in connection with chattering avoidance. The related output feedback sliding mode (OFSM) control scheme based on asymptotic state observers was analyzed. However, the plant was assumed known except for singular perturbations representing fast dynamics of sensors and actuators.

In the late 1980's, several papers about OFSM control of uncertain plants started to appear. Since then, OFSM stabilization control strategies for plants of relative degree one, i.e., $\mathrm{rank}\{C_p B_p\} = n$, have been considered by several authors [44, 10] (see also references therein). In this case, static output feedback is possible [46]. In [18], an early version of the UV-MRAC controller, named VS-MRAC (Variable Structure Model Reference Adaptive Control) was first proposed as a solution for global output tracking of uncertain *linear* SISO plants with relative degree one by output-feedback. The generalization of the VS-MRAC to uncertain linear plants of arbitrary relative degree appeared in [17, 21] (see also references therein) and an application to the control of a flexible spacecraft can be founded in [47]. In [6], an extension of the VS-MRAC to the MIMO case was developed. More recently, the UV-MRAC was introduced in [20] and can be regarded as an alternative extension of the VS-MRAC for MIMO linear plants which requires weaker prior knowledge of the high frequency gain matrix K_p, thanks to the use of unit vector control instead of the "sign($\cdot$)" vector control as in [6].

Early papers on OFSM control for nonlinear systems with arbitrary relative degree, were based on high-gain observers (HGO) [12, 11, 35, 36]. Combined with saturation, HGO leads to a kind of separation principle (see [27, 2]), which makes it very appealing for general nonlinear systems. While HGO has been widely used in the literature during the last decade, it is well known that it can lead to undesirable large transients known as the *peaking phenomena*. This is linked to the fact that high gain observers are essentially filters that estimate signal derivatives up to a certain order. Intentional saturation of the control signal can attenuate peaking, however, global stability is lost, in general. It is interesting to note that the application of the HGO to *linear* systems also leads to peaking. Recently, in [9], a peaking free and globally stable OFSM control was proposed for linear SISO systems based on HGO, without the need of control saturation. An alternative promising trend for OFSM control is to use higher order sliding modes and exact robust differentiators [14], [4], [30]. Here also, the existing (truly) output feedback based strategies do not lead to global stability properties, as a rule.

Within the context of VS-MRAC, nonlinear SISO plants started to be considered in [33]. The resulting OFSM controller was not based on explicit HGO and was free of peaking. Later on, a generalization to MIMO linear plants with relative degree one and under the action of a class of locally (not necessarily globally) Lipschitz nonlinear disturbances was presented in [19], where global exponential stability was demonstrated.

This chapter generalizes the UV-MRAC, described in [19], to the more challenging problem of MIMO plants with arbitrary relative degree. The class of plants considered here is formulated as in several other works, e.g. [11, 10]. It consists of linear plants under the action of matched or unmatched nonlinear disturbances. Previous work in this direction were mostly based on matched disturbances which should, by assumption, be norm-bounded by some function of the plant measured output. The more challenging problem of unmatched disturbances is included in the class considered here where mismatch is allowed even for a class of disturbances depending on the unmeasured states.

Note that, to the best of the authors' knowledge, no global stability result has yet been achieved by OFSM control for uncertain systems with relative degree higher than one, without the assumption of global Lipschitz condition on the plant nonlinearities. In this case, only semi-global stability can be guaranteed. Unfortunately, this is also true for the UV-MRAC described in this chapter. On the other hand, output feedback global regulation or tracking for a subclass of such systems (locally Lipschitz) can be achieved by other nonlinear control designs [32], [1], [28].

Thus, the design of global regulation or tracking OFSM controllers for such category of plants seems to represent a particularly challenging and interesting problem, even in the case of SISO systems. A positive answer could be expected when it is possible to design an output feedback controller for global regulation or tracking, under the assumption that the plant is completely known.

Among the topics for future research one could point out: the experimental evaluation of the UV-MRAC strategy, the related sensitivity analysis with respect to measurement noise, the inclusion of more general nonlinear systems, the extension of the peaking free OFSM controller based on HGO introduced in [9] to nonlinear plants, and the combination of the UV-MRAC with exact robust differentiators based on higher order sliding modes for exact output tracking, as in [34].

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Appendix: Proof of Theorem 3

In what follows, k_i denote positive constants, $\Psi_i(\cdot)$ denote functions of class $\mathcal{K}_\infty$ and $C(t)$ is a generic term of the form given in (66), with an appropriate function $\Psi_X \in \mathcal{K}_\infty$ and constant k_{red}. From (29), (22) and (25) one has

$$\|(x_p)_t\|_\infty, \|\omega_t\|_\infty \le k_1\|(X_e)_t\|_\infty + k_2 \,. \tag{95}$$

From Assumptions (A6)–(A8), (81) holds. Now, from (95) and the fact that U_N also satisfies the inequality in Assumption (A8), then $\bar{U}$, given in (50), can be bounded by

$$\|\bar{U}(t)\| \le C(t) \,. \tag{96}$$

Now, it is convenient to rewrite (54) as

$$\varepsilon_N = L_N^{-1}[-U_N - \bar{U}] + \bar{\beta}_{uN} - \pi_{eN} - \pi_{0N} \,, \tag{97}$$

$$\bar{\beta}_{uN} = (F_{1,N} - I)F_{1,N}^{-1}L_N^{-1}\bar{U} \,. \tag{98}$$

Since the norm of the operator in (98) is of order $\mathcal{O}(\tau)$, then $\|(\bar{\beta}_{uN} - \bar{\beta}_{uN}^0)_t\|_\infty \le \tau\bar{k}_{\beta N}\|(\bar{U})_t\|_\infty \le \tau\bar{k}_{\beta N}C(t)$, where the last inequality comes from (96) and $\bar{k}_{\beta N}$ is a positive constant.

Recalling that $u = u^{\rm nom} - S_p U_N$, one notes that L_N^{-1} in (97) operates on the same signal as the one in (31). From (31), (32), (50) and (54), the model following error can be rewritten as: $\dot{X}_e = A_c X_e + B_c K^{\rm nom}\left[\dot{\hat{e}}_N + \alpha_N \hat{e}_N\right]$, where $\hat{e}_N := \varepsilon_N - (\bar{\beta}_{uN} - \pi_{eN} - \pi_{0N})$. To eliminate the derivative term $\dot{\hat{e}}_N$, a variable transformation $\bar{X}_e := X_e - B_c K^{\rm nom}\hat{e}_N$ is performed yielding

$$\dot{\bar{X}}_e = A_c\bar{X}_e + (A_c + \alpha_N I)B_c K^{\rm nom}\hat{e}_N \,. \tag{99}$$

Applying Theorem 2, the bound (63) and the exponential stability of A_c imply that $\bar{X}_e(t)$ is bounded by $\|\bar{X}_e(t)\| \le \tau C(t) + \Pi$, where Π is a generic class $\mathcal{KL}$ function represented here by $\mathcal{V}_1(\|z(0)\|, t)$ (see Property 1). Similarly,

$$\|X_e(t)\| \le \tau C(t) + \Pi \,. \tag{100}$$

Let the generic $C(t)$, defined in (66), be specified here as

$$C(t) := \Psi_1(\|(X_e)_t\|_\infty) + k_1 \,. \tag{101}$$

From (100) one has

$$\|(X_e)_t\|_\infty \le \tau C(t) + \Psi_2(\|z(0)\|) \,, \tag{102}$$

where $\Psi_2(\|z(0)\|) = \mathcal{V}_1(\|z(0)\|, 0)$ comes from the initial value of the term Π appearing in the bound (100). Thus $C(t)$, defined in (101), is bounded by

$$C(t) \le \Psi_1(\tau C(t) + \Psi_2(\|z(0)\|)) + k_1 \,. \tag{103}$$

Applying the separability Property 2 to (103) results in

$$C(t) \le \Psi_1(\tau(\alpha + 1)C(t)) + \Psi_3(\|z(0)\|) + k_1 \,, \tag{104}$$

where α is an arbitrary positive constant. Now, given $R > 0$ and $0 < R_0 < R$, then for some $t^* \in (0, t_M)$ and $\|z(0)\| < R_0$ one has $\|z(t)\| < R$ for $t \in [0, t^*)$.

Assume $t \in [0, t^*)$. According to Assumptions (A5) and (A7), the function Ψ_1 can be chosen locally Lipschitz. In addition, while $t \in [0, t^*)$, the complete error state z is bounded by the arbitrary constant R. Thus, from (104), one has $\Psi_1(\tau(\alpha+1)C(t)) \leq \tau k(R)C(t)$, where $k(\cdot) = k_0 + \psi(\cdot)$, with $\psi(\cdot)$ being of class $\mathcal{K}$. For a given R, $k(R)$ is a positive constant which increases as $R \to +\infty$, but is not necessarily unbounded. Thus, from (104), the following upper bound holds

$$C(t) \leq \tau k(R)C(t) + \Psi_3(\|z(0)\|) + k_1 \,. \tag{105}$$

Now, from (105), after simple algebraic manipulation one obtains

$$C(t) \leq \frac{\Psi_3(\|z(0)\|) + k_1}{1 - \tau k(R)} := \Psi_4(\|z(0)\|) + \frac{k_1}{1 - \tau k(R)} \,, \tag{106}$$

which is valid for $\tau < 1/k(R)$. Applying (106) in (100) it follows

$$\|X_e(t)\| \leq \tau \Psi_4(\|z(0)\|) + \mathcal{O}(\tau) + \Pi \,. \tag{107}$$

Now, with the partition $z^T = [(z^0)^T, z_e^T]$, where $z_e^T := [\varepsilon_N^T, X_e^T]$ and from (107) and the bound (63) in Theorem 2, one gets

$$\|z(t)\| \leq \mathcal{V}(\|z(0)\|, t) + \tau \Psi_5(\|z(0)\|) + \mathcal{O}(\tau) \,. \tag{108}$$

where $\mathcal{V}(\|z(0)\|, t) \in \mathcal{KL}$ is obtained from the Π terms in (107) and (63) and considering the fact that the variables X_ε, ε_i $(i = 0, \ldots, N-1)$ and $x_{\mathcal{F}}^0$ are bounded by Π^0 in Theorem 2.

Let $\Psi_6(\|z(0)\|) := \mathcal{V}(\|z(0)\|, 0) + \tau \Psi_5(\|z(0)\|)$. Since $\Psi_6 \in \mathcal{K}_\infty$ one can define $R_0 := \Psi_6^{-1}(R - \mathcal{O}(\tau))$, for $R > \mathcal{O}(\tau)$. Now, since $z(t)$ is absolutely continuous, $\|z(0)\| \leq R_0$ implies $\|z(t)\|$ is bounded away from R as $t \to t^*$. If one assumes that t^* is finite then $\|z(t)\| < R - \varepsilon_R$, $\forall t < t^*$ and some constant $\varepsilon_R > 0$. Therefore, one cannot reach the boundary of $B_R = \{z : \|z(t)\| < R\}$ in finite time. This implies that $z(t)$ is uniformly bounded and cannot escape in finite time, i.e., $t_M = +\infty$. Furthermore, the constant R_0 can be made arbitrarily large when R increases and $\tau \to +0$.

From Assumption (A8) and (95), U_N and ω are also uniformly bounded. Thus, U_{av}, $\hat{X}$ and ϱ_i $(i = 0, 1, \ldots, N-1)$ are all bounded. Consequently, the states of any (stable) realizations of the averaging filters and the filters appearing in (79), (80), (85)–(89), are also uniformly bounded.

Noting that the initial time is irrelevant in deriving (108), then one has, for arbitrary $t_k \geq 0$ and $t \geq t_k$

$$\|z(t)\| \leq \mathcal{V}(\|z(t_k)\|, t - t_k) + \tau \Psi_5(\|z(t_k)\|) + \mathcal{O}(\tau), \quad \forall t \geq t_k \,. \tag{109}$$

Now, any $\mathcal{V} \in \mathcal{KL}$ can be bounded by $\mathcal{V}((\cdot), t) \leq \Psi_7(\cdot)\nu(t)$, with $\Psi_7 \in \mathcal{K}_\infty$ and $\nu(t) \in \mathcal{L}$ [39][proof of Lemma 8]. In addition, from Assumptions (A6)-(A7), $\Psi_5(\sigma), \Psi_7(\sigma) \in \mathcal{K}_\infty$ can be chosen locally Lipschitz. Thus, for a given $\sigma_0 := \|z(t_0)\|$, (109) implies $\|z(t)\| \leq \bar{\sigma}(\sigma_0)$ where $\bar{\sigma}(\cdot)$ is a positive increasing function. Thus, there exists a positive increasing function $\kappa(\sigma_0)$ similar to $k(R)$, such that $\Psi_5(\sigma) \leq \kappa(\sigma_0)\sigma$ and $\Psi_7(\sigma) \leq \kappa(\sigma_0)\sigma$, $\forall\sigma \leq \bar{\sigma}(\sigma_0)$. Recalling that, $\|z(t)\| \leq \bar{\sigma}(\sigma_0), \forall t$, then

$$\|z(t)\| \leq \kappa(\sigma_0)\,[\nu(t-t_k)+\tau]\,\|z(t_k)\| + \mathcal{O}(\tau)\,, \quad \forall t \geq t_k\,. \tag{110}$$

Since $\nu \in \mathcal{L}$, then from (110), for $\tau\kappa(\sigma_0) < 1$, there exists a large enough constant $T > 0$ such that $\lambda := \kappa(\sigma_0)(\nu(T)+\tau) < 1$, and thus, for the state samplings at $t_0, t_1, \ldots$, defined by $t_{k+1} = t_k + T, \quad (k = 0, 1, \ldots)$, one has a simple linear recursion

$$\|z(t_{k+1})\| \leq \lambda\|z(t_k)\| + \mathcal{O}(\tau)\,. \tag{111}$$

Therefore, by continuity of the solutions $z(t)$ with respect to initial conditions $z(t_k)$, one concludes that, for τ small enough, the error system is semi-globally exponentially stable with respect to a residual set of order τ which is independent of the initial conditions. As a matter of fact, from (110) and (111) one can establish the following inequality

$$\|z(t)\| \leq e^{-a(\|z(0)\|)\,t}\,\Psi_8(\|z(0)\|) + \mathcal{O}(\tau), \quad \forall t \geq 0\,, \tag{112}$$

where $a(\cdot) > 0$ and $\Psi_8(\cdot) \in \mathcal{K}_\infty$.

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