First Order Upperbound for Norms of Filtered Signals

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Abstract

This report considers the design of nonlinear filters applied in variable structure and sliding mode control systems. The focus is on the problem of approximating an upperbound for the norm of a filtered vector signal by through a first order filter. Two approximation methods are described: (i) a Lyapunov based scheme which is simple but may result in conservative upperbounds and (ii) an optimization method which produces the smallest upperbound.

Keywords — nonlinear filter, norm of a signal, norm upperbound, filter approximation, variable structure control, sliding mode control, non-convex optimization.

1 Introduction

This report considers the design of nonlinear filters applied in variable structure and sliding mode control systems such as (Emelyanov, Korovin, Nersisian & Nisenzon 1992, Hsu, Lizarralde & Araújo 1997, Hsu, Cunha, Costa & Lizarralde 2002). The focus is on the problem of approximating an upperbound for the norm of a filtered vector signal by the signal generated by a first order filter, i.e., the synthesis of the parameters c_1 and γ such that

$$\|g(t) * u(t)\| \leq c_1 e^{-\gamma t} * \|u(t)\|\,,\ \forall t \geq 0\,, \tag{1}$$

for a given filter of strictly proper rational $p \times m$ transfer function matrix $G(s)$ and impulsive response $g(t)$. The input signal u is assumed to be piecewise continuous or a switched control generated by a sliding mode control system. The first order approximation filter (FOAF) has the nonlinear realization

$$\dot{\bar{y}}(t) = -\gamma \bar{y}(t) + c_1 \|u(t)\| \,, \tag{2}$$

where $\bar{y}(t) \geq \|g(t) * u(t)\|$, $\forall t \geq 0$, provided that $\bar{y}(0) \geq 0$.

2 Preliminaries

The following notation and basic concepts are employed in this report.

- The maximum and minimum eigenvalues of a symmetric matrix P are denoted as $\lambda_{max}(P)$ and $\lambda_{min}(P)$, respectively.
- $\|x\|$ denotes the Euclidean norm of a vector x and $\|A\| = \sigma_{max}(A)$ denotes the corresponding induced norm of a matrix A.
- *The stability margin of a $p \times m$ rational transfer function matrix $G(s)$* is given by

  $$\gamma_0 := \min_j \{-Re(p_j)\} \,, \tag{3}$$

  where $\{p_j\}$ are the the poles of $G(s)$. If $G(s)$ is bounded-input-bounded-output (BIBO) stable, then $\gamma_0 > 0$.
- *The stability margin of a square real matrix A* is given by

  $$\lambda_0 := \min_j \{-Re(\lambda_j)\} \,, \tag{4}$$

  where $\{\lambda_j\}$ are the the eigenvalues of A. If $\lambda_0 > 0$, then A is said to be Hurwitz.
- Filippov's definition for the solution of discontinuous differential equations is assumed (Filippov 1964).
- A function $f(t)$ ($f : \mathbb{R} \to \mathbb{R}$) is *unimodal* on $[a, b]$ if there is a point $t^* \in [a, b]$ with the property that f is strictly monotone decreasing on $[a, t^*]$ and strictly monotone increasing on $[t^*, b]$. This is a derivative-free way of saying that f has a unique global minimum at t^* (Van Loan 2000).

3 Approximations for The Norm of a Filtered Signal

The theorem presented below is fundamental for the approximation techniques discussed in this report. This theorem follows Lemma 3.1 of (Ioannou & Tsakalis 1986) which was developed for the scalar case.

Theorem 1 *Consider the system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,, \\ y(t) &= Cx(t)\,, \end{aligned} \tag{5}$$

where $y \in \mathbb{R}^p$*,* $u \in \mathbb{R}^m$ *and* $x \in \mathbb{R}^n$*. The system transfer function matrix is given by* $G(s) = C(sI - A)^{-1}B$*. Let* γ_0 *be the stability margin of* $G(s)$ *and* $\gamma := \gamma_0 - \delta$ *with* $\delta > 0$ *being an arbitrary constant. Let* $\bar{u}(t)$ *be an instantaneous upperbound of* $u(t)$*, i.e.,* $\|u(t)\| \leq \bar{u}(t)$*,* $\forall t \geq 0$*. Then,* $\exists c_1, c_2 > 0$ *such that the impulse response* $g(t)$ *satisfies* $\|g(t)\| \leq c_1 \exp(-\gamma t)$ *and the following inequalities hold*

$$\|g(t) * u(t)\| \leq c_1 \exp\{-\gamma t\} * \bar{u}(t)\,, \tag{6}$$

$$\|y(t)\| \leq c_1 \exp\{-\gamma t\} * \bar{u}(t) + c_2 \exp\{-(\lambda_0 - \delta)t\}\|x(0)\|\,, \tag{7}$$

$\forall t \geq 0$*, where* λ_0 *is the stability margin of* A*.*

Proof: The system impulse response is given by

$$g(t) = C \exp\{At\}B \qquad t \geq 0. \tag{8}$$

The elements of the matrix exponential $\exp\{At\}$ and of the impulsive response $g(t)$ can be represented as sums of exponential terms $A_{jl}t^l \exp\{\lambda_j t\}$ and $G_{jl}t^l \exp\{\lambda_j t\}$, respectively, where $0 \leq l \leq m_j - 1$, m_j is the multiplicity of the eigenvalue λ_j of A, $A_{jl} \in \mathbb{C}^{n\times n}$ and $G_{jl} \in \mathbb{C}^{p\times m}$. Thus we conclude that $\exists c_1, c_3 > 0$ such that $\|\exp\{At\}\| \leq c_3 \exp\{-(\lambda_0 - \delta)t\}$, $\|g(t)\| \leq c_1 \exp\{-\gamma t\}$, $\forall t \geq 0$ and

$$\|g(t) * u(t)\| \;\leq\; \|g(t)\| * \|u(t)\| \leq c_1 \exp\{-\gamma t\} * \bar{u}(t)\,. \tag{9}$$

In the other hand the output signal of the filter (5) is given by

$$y(t) = g(t) * u(t) + C \exp\{At\}x(0)\,, \tag{10}$$

which let us conclude that $\|y(t)\|$ can be bounded by the sum of the upperbound (6) and $\|C\|c_3 \exp\{-(\lambda_0 - \delta)t\}\|x(0)\|$. Choosing $c_2 = \|C\|c_3$ we finally get (7), thus completing the proof. ■

Remark 1 *Since the system may be non controllable or non observable some eigenvalues may not be poles of* $G(s)$*, then the stability margins obey* $\lambda_0 \leq \gamma_0$*. Moreover, if the autonomous system is asymptotically stable, then* $0 < \lambda_0 \leq \gamma_0$*.*

Remark 2 *If the system output equation is* $y(t) = Cx(t) + Du(t)$*, then the transfer function matrix is given by* $G(s) = C(sI - A)^{-1}B + D$ *and Theorem 1 holds with the addition of the term* $\|D\|\bar{u}(t)$ *to the bounds (6) and (7). Of course the impulse response* $g(t)$ *is also modified.*

3.1 Filter for discontinuous signals

Consider the class of nonlinear systems described by (Utkin 1978)

$$\dot{x}_p = f(x_p, t) + \Gamma(x_p, t)u\,, \tag{11}$$

where $f(\cdot)$ and $\Gamma(\cdot)$ are smooth vector fields and x_p is the state vector. The control signal is generated through some sliding mode control law such as the unit vector control law of the form (Hsu et al. 2002)

$$u = -\rho(x_p, t)\frac{v(x_p, t)}{\|v(x_p, t)\|}\,, \tag{12}$$

where $v(x_p, t)$ is a vector function of the state of the system and $\rho(x_p, t) \geq 0$, $\forall x$, $\forall t$. We refer to $\rho(\cdot)$ as the unit vector modulation function, which is designed to induce a sliding mode on the manifold $v(x_p, t) = 0$. We will henceforth assume that $u = 0$ if $v(x_p, t) = 0$ only to have a complete definition of the control law. Let $x_p(t)$ be a solution of this system, for $t \in [0, T)$. Then, the extended equivalent control is a function locally integrable in the sense of Lebesgue, defined almost everywhere in the interval $[0, T)$, and is given by

$$u_{eq} = -[J\Gamma(x_p(t), t)]^{-1}[Jf(x_p(t), t) + \frac{d}{dt}v(x_p(t), t)]\,, \tag{13}$$

where $J = \frac{\partial}{\partial x_p}v(x_p(t), t)$. The above expression is well defined since the solution $x_p(t)$ is absolutely continuous by definition and, thus, has derivatives almost everywhere.

Certain control systems require an upperbound for the filtered control signal $\|g(t) * u\|$ (Hsu et al. 2002). A solution is to apply Theorem 1 directly, which leads to $\|g(t) * u\| \leq c_1 \exp\{-\gamma t\} * \|u\|$. However, this would result in a conservative estimate, since u involves discontinuous terms and $\|u\| = \rho(x_p, t)$ as can be concluded from (12). A better alternative would be to use the equivalent control (13) since $\|g(t) * u\| = \|g(t) * u_{eq}(t)\|$ and $\|u_{eq}(t)\| \leq \rho(x_p, t)$. However, the equivalent control is unavailble but can be approximated by the average control generated by the low-pass filter

$$u_{av} = \frac{I}{\tau s + 1}u\,, \tag{14}$$

with sufficiently small time constant $\tau > 0$ and unit DC gain, then $\|g(t) * u\| \approx \|g(t) * u_{av}(t)\|$. The scheme of (Min & Hsu 2000, Proposition 1) applies the average control but overcomes approximations and does not require that τ be small. This proposition is revised in Theorem 2 with two minor corrections: (i) the constant c_2 was missing *loc. cit.* and (ii) here $\gamma := \gamma_0 - \delta$ ($\delta > 0$) replaces $\gamma := \gamma_0$ in Propositon 1 which is incorrect if $G(s)$ has multiple poles with real part equal to $-\gamma_0$, since $\nexists c \in \mathbb{R}$ such that $c\exp\{-\gamma_0 t\} \geq t^l \exp\{-\gamma_0 t\}$, $\forall t \geq 0$, $l \geq 1$.

Theorem 2 *Consider the system (5) with transfer function matrix $G(s) = C(sI - A)^{-1}B$. Let γ_0 be the stability margin of $G(s)$ and $\gamma := \gamma_0 - \delta$ with $\delta > 0$ being an arbitrary constant.*

Let $u_{av}(t)$ be the average control generated by the filter (14) with arbitrary time constant $\tau > 0$. Then, $\exists c_1, c_2, c_3 > 0$ such the following inequalities hold

$$\|g(t) * u(t)\| \leq c_1 \exp\{-\gamma t\} * \|u_{av}(t)\| + \tau c_2 \|u_{av}(t)\| \,, \tag{15}$$
$$\|y(t)\| \leq c_1 \exp\{-\gamma t\} * \|u_{av}(t)\| + \tau c_2 \|u_{av}(t)\| + c_3 \exp\{-(\lambda_0 - \delta)t\} \|x(0)\| \,, \tag{16}$$

$\forall t \geq 0$, where λ_0 is the stability margin of A.

Proof: Lets write the identity

$$G(s)u \;=\; (\tau s + 1)G(s)\frac{I}{\tau s + 1}u_{eq} = (\tau s + 1)G(s)u_{av} \,, \tag{17}$$

which can be rewritten in terms of the state space matrices (5) as

$$\begin{aligned} G(s)u &= (\tau s + 1)C(sI - A)^{-1}Bu_{av} = C(\tau s + 1)(sI - A)^{-1}Bu_{av} \\ &= C(\tau A + I)(sI - A)^{-1}Bu_{av} + \tau CBu_{av} \,. \end{aligned} \tag{18}$$

From (18) we conclude that

$$\|G(s) * u\| \;\leq\; \|C(\tau A + I)\exp\{At\}B * u_{av}\| + \|\tau CBu_{av}\| \,, \tag{19}$$

which uppon the application of Theorem 1 to the filter $C(\tau A + I)(sI - A)^{-1}B$ and choosing $c_2 \geq \|CB\|$ results in inequality (15). Inequality (16) is obtained by the addition of an exponentially decaying term due to the initial conditions of the filter (5), analogously to the proof of Theorem 1. ■

Remark 3 *The computation of the constant c_2 in Theorem 2 can be carried out through the application of the identity*

$$CB = \lim_{s \to \infty} sG(s) \,, \tag{20}$$

which is valid since $G(s)$ is strictly proper by assumption.

4 Filter Synthesis

Theorem 1 provides existence conditions for the constants c_1 and c_2 and an upperbound for γ which is adequate for the stability analysis of control systems (e.g., (Hsu et al. 2002)) but is insufficient for design. In the design of sliding mode control systems, the values for the constants c_1 and γ should be selected to keep signal amplitudes at moderate levels.

Hereon, the realization $\{A, B, C\}$ is assumed minimal and stable, which implies $\gamma_0 = \lambda_0 > 0$.

4.1 Performance criteria

The synthesis of the FOAF should be based on some criteria oriented to the filter application. Three features of the FOAF are considered here:

1. Filter dynamics velocity given by γ.

2. The DC gain of the filter.

3. Peak value of the filter impulse response. This value is equal to c_1.

If the application requires a fast response, then the natural criteria is to choose a large γ, restricted by the bounds

$$\gamma \leq \gamma_0 - \delta_u \,, \tag{21}$$

imposed by Theorem 1 with $\delta \geq \delta_u > 0$, and

$$\gamma \geq \delta_l \,, \tag{22}$$

since the FOAF should have some specified stability margin $\delta_l > 0$.

Applications which cannot tolerate large peaks in response to fast input pulses, require that the constant c_1 should be as small as possible, subject to the constraint

$$c_1 \exp\{-\gamma t\} \geq \|g(t)\| \,, \qquad \forall t \geq 0 \,, \tag{23}$$

which is equivalent to

$$c_1 \geq \sup_{0 \leq t < +\infty} \left\{ \frac{\|g(t)\|}{\exp\{-\gamma t\}} \right\} . \tag{24}$$

The DC gain of the FOAF is an important criteria in view of the following aspects: **(i)** the DC gain coincides with the area of the first order filter impulse response and **(ii)** the DC gain should be low to keep small the amplitude of sliding mode control system modulation function generated by the FOAF. The DC gain is given by

$$S = \int_0^{+\infty} c_1 \exp\{-\gamma t\} dt = \frac{c_1}{\gamma} \, . \tag{25}$$

4.2 Lyapunov synthesis

In this section we present a simple synthesis technique based on Lyapunov quadractic forms, which can be computed through well stablished numeric algorithms (see (Patel, Laub & Van Dooren 1994, Part 7)). Since the matrix A of the filter realization (5) is Hurwitz by assumption, then a positive definite solution $P = P^T$ for the Lyapunov equation $A^T P + PA = -2Q < 0$ exists. The best estimate for the stability margin of A (λ_0) using the Lyapunov equation is achieved with $Q = I$ and is equal to $\lambda_L = 1/\lambda_{max}(P)$ (see (Edwards & Spurgeon 1998, pp. 22-23)). The fact that $0 < \lambda_L \leq \lambda_0$ can be concluded from (Van Loan 1985, Theorem 1.1).

4.2.1 Coefficients computation

To compute the constants c_1, c_2 and γ of (6) and (7), we apply the quadractic form $V(x) = x^T P x$ which has time derivative bounded by

$$\begin{aligned} \dot{V} &= -2\|x\|^2 + 2x^T PBu \\ &\leq -2\|x\|^2 + 2\|PB\|\,\|x\|\,\|u\|\,. \end{aligned} \tag{26}$$

Applying the Rayleigh-Ritz inequality

$$\lambda_{min}(P)\,\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\,\|x\|^2\,, \tag{27}$$

we get

$$\dot{V} \leq -\frac{2}{\lambda_{max}(P)}V + \frac{2\|PB\|}{\sqrt{\lambda_{min}(P)}}\sqrt{V}\|u\|\,. \tag{28}$$

Then, defining $r := \sqrt{V}$, one obtains

$$\dot{r} \leq -\frac{1}{\lambda_{max}(P)}r + \frac{\|PB\|}{\sqrt{\lambda_{min}(P)}}\|u\|\,. \tag{29}$$

An upper bound $\bar{r}(t)$ of $r(t)$, is given by the solution of

$$\dot{\bar{r}} = -\frac{1}{\lambda_{max}(P)}\bar{r} + \frac{\|PB\|}{\sqrt{\lambda_{min}(P)}}\|u\|\,, \qquad \bar{r}(0) = r(0) \geq 0\,. \tag{30}$$

The solution of the equation above is

$$\bar{r}(t) = \frac{\|PB\|}{\sqrt{\lambda_{min}(P)}}\exp\{-\lambda_L t\} * \|u(t)\| \ + \exp\{-\lambda_L t\}r(0)\,, \qquad t \geq 0\,. \tag{31}$$

Uppon the application of the Rayleigh-Ritz inequality (27) to (31) and remembering that $\|y(t)\| \leq \|C\|\|x(t)\|$ we conclude that

$$\|y(t)\| \leq \frac{\|C\|\,\|PB\|}{\lambda_{min}(P)}\exp\{-\lambda_L t\} * \|u(t)\| \ + \|C\|\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\exp\{-\lambda_L t\}\|x(0)\|\,. \tag{32}$$

Finally, this Lyapunov scheme results in the following estimates for the constants in (6) and (7):

$$\gamma = \frac{1}{\lambda_{max}(P)}\,, \qquad c_1 = \frac{\|C\|\,\|PB\|}{\lambda_{min}(P)}\,, \qquad c_2 = \|C\|\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\,. \tag{33}$$

4.2.2 Lyapunov synthesis example

We examine the Lyapunov synthesis of a first order approximation filter for the norm of the output signal of a filter which transfer function is given by

$$G_1(s) = \frac{10}{(s+1)(s+2)} \,. \tag{34}$$

To evaluate Lyapunov synthesis dependency on the filter realization, we consider three realizations on Table 1: controllable, cascade and parallel. The impulse responses of each first order approximation filter and the norm of the impulse response of $G_1(s)$ ($|g_1(t)|$) are presented in Figure 1. The Lyapunov synthesis results on Table 1 show that the

Table 1: Lyapunov synthesis results for three realizations of $G_1(s)$.

	Realization matrices			Lyapunov synthesis results		
Realization	A	B	C	γ (s^{-1})	c_1	DC gain
Controllable	$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 10 & 0 \end{pmatrix}$	0.38	18.6	48.5
Cascade	$\begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 10 & 0 \end{pmatrix}$	0.82	16.2	19.6
Parallel	$\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 10 & -10 \end{pmatrix}$	1.00	31.7	31.7

parrallel realization has the fastest dynamics and it attains the largest admissible estimate for γ. This is justified by the fact that the matrix A is diagonal. Then the solution of the Lyapunov equation is $P = -A^{-1}$, which gives the best estimate for the stability margin $(1/\lambda_{max}(P) = \gamma)$. However, in this example the parallel realization produces the larger constant c_1, consequently its impulse response has the largest peak value, as can be seen in Figure 1. It is also remarkable that the cascade realization has the smallest peak value and DC gain.

A common illness of the Lyapunov sysnthesis through any of these approximations is that the parameters of the approximation filter are over estimated. This becomes clear in the comparison of the impulse response of each approximation filter with the norm of the filter impulse response, as can be seen in Figure 1.

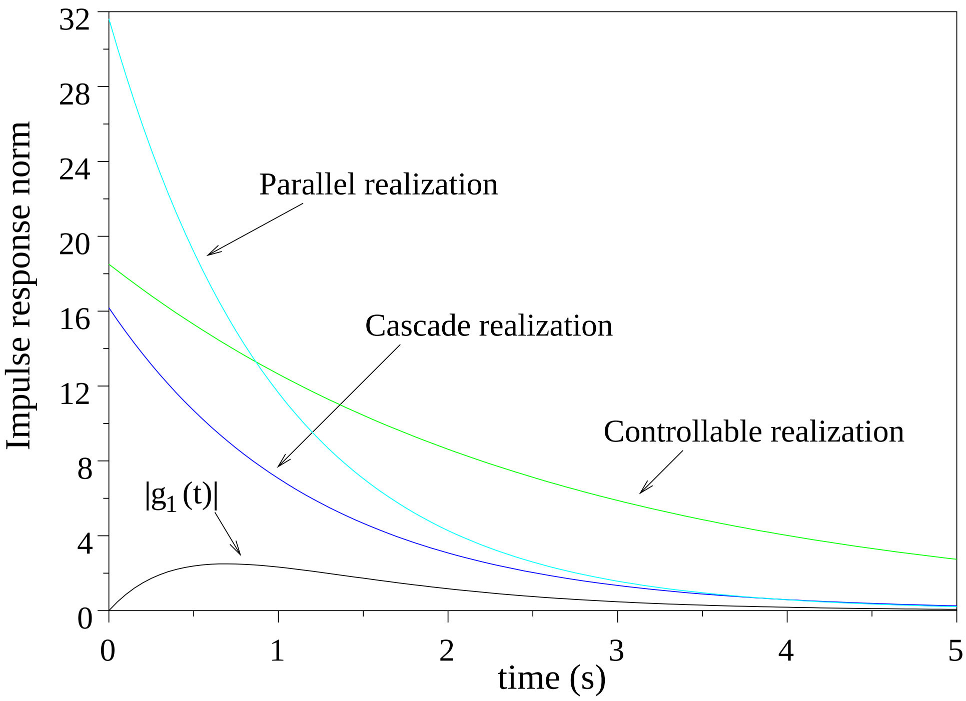

Figure 1: Impulse response of each first order approximation filter obtained through Lyapunov synthesis and the norm of the impulse response of $G_1(s)$ ($|g_1(t)|$).

4.3 Optimization of filter coefficients

The design of the FOAF can be formulated as an optimization problem restricted to the feasible set defined by the bounds in Section 4.1:

$$\begin{aligned} \mathcal{V} = \{\, (c_1,\gamma)\ :\ c_1,\gamma &\in \mathbb{R}^+\,, && (35)\\ c_1 &\geq C_1(\gamma)\,, && (36)\\ \gamma &\leq \gamma_0-\delta_u\,, && (37)\\ \gamma &\geq \delta_l\,\}\,, && (38) \end{aligned}$$

where

$$C_1(\gamma) := \sup_{0\leq t<+\infty} \left\{ \frac{\|g(t)\|}{\exp\{-\gamma t\}} \right\} . \tag{39}$$

Three optimization problems are stated in what follows.

1. *Maximize the filter dynamics velocity γ and keep small the peak value of the filter impulse response c_1.* The maximum velocity is $\gamma = \gamma_0 - \delta_u$, as can be concluded from

Theorem 1 and the constraint (37), thus the solution of this problem is given by

$$c_1 = C_1(\gamma_0 - \delta_u)\,, \tag{40}$$

$$\gamma = \gamma_0 - \delta_u\,. \tag{41}$$

2. *Minimize the filter DC gain.* This problem can be stated as

$$\text{minimize} \quad S(c_1, \gamma) = \frac{c_1}{\gamma}\,, \tag{42}$$

$$\text{subject to} \quad (c_1, \gamma) \in \mathcal{V}\,. \tag{43}$$

The resolution of this problem can be simplified through the substitution of the inequality constraint (36) by the equality constraint $c_1 = C_1(\gamma)$, since for a fixed γ we can assert that S is minimum if and only if c_1 is also minimum.

3. *Minimize the peak value of the filter impulse response.* This problem can be stated as

$$\text{minimize} \quad c_1\,, \tag{44}$$

$$\text{subject to} \quad (c_1, \gamma) \in \mathcal{V}\,. \tag{45}$$

The resolution of this problem can be simplified through the substitution of the inequality constraint (36) by the equality constraint $c_1 = C_1(\gamma)$, as is done in the previous optimization problem. Moreover, generally the cost function (44) decreases as γ decreases, then the inequality restrictions (37) and (38) can be replaced by the simpler equality restriction $\gamma = \delta_l$.

4.3.1 Estimation of the *supremum*

A key problem found in the three optimization problems stated above is the computation of the *supremum* $C_1(\gamma)$ defined in (39). The computation of the *supremum* solves the optimization problems 1 and 3 directly. However, the main difficulties are:

1. The function $\|g(t)\| \exp\{\gamma t\}$ may have an infinite number of *local minima* over the interval $t \in [0, +\infty)$. Thus the computation of $C_1(\gamma)$ is generally a non convex optimization problem.

2. The function $\|g(t)\|$ may be non analytic over all $t \in [0, +\infty)$, which is a difficulty in the application of derivative based optimization methods.

3. The search horizon is unconstrained.

To avoid these difficulties, the following simplifications where adopted:

(i) The adopted search horizon is finite, i.e., $t \in [0, t_f]$, $0 < t_f < +\infty$. This is fundamented on the BIBO stability of $G(s)$, which implies that $g(t) \to 0$ exponentially as $t \to +\infty$. Thus t_f should be chosen such that $\|g(t)\| \exp\{\gamma t\}$ be negligible $\forall t \geq t_f$.

(ii) The *maximum* is search in a finite number of periodic samples in the interval $t \in [0, t_f]$. The *sampling interval* $h \in \mathbb{R}^+$ should be chosen such that all the *local maximum* can be computed with an acceptable accuracy.

Finally, the estimated value for $C_1(\gamma)$ is given by

$$\hat{C}_1(\gamma, h, k_f) := \max_{k \in [0,\ldots,k_f]} \left\{ \frac{\|\hat{g}(kh)\|}{\exp\{-\gamma k h\}} \right\} , \tag{46}$$

where $k, k_f \in \mathbb{N}$, $t_f = k_f h$ and $\hat{g}(kh)$ is the periodically sampled impulse response.

Impulse response computation

The sampled impulse response can be computed through the recursion formula below which avoids the computation of the matrix exponential in each step:

$$\begin{aligned} X(kh+h) &= \exp\{Ah\} X(kh) , \qquad X(0) = B , \\ \hat{g}(kh) &= C X(kh) , \end{aligned} \tag{47}$$

where for convinience $\hat{g}(0) = g(0+)$ and $\hat{g}(kh) = g(kh)$ for $k \geq 1$. An analogous recursion can be applied to the computation of $\exp\{-\gamma k h\}$.

The folowing *rules of thumb* based on the natural frequencies of $G(s)$ can be applied in the choice of the parameters h and k_f:

$$h = \frac{\varepsilon_1}{\max_j\{|\lambda_j|\}} , \tag{48}$$

$$k_f = \text{int}\left(0.5 + \frac{h_o}{h \min_j\{|\lambda_j|\}}\right) , \tag{49}$$

where $\{\lambda_j\}$ are the eigenvalues of A, $\varepsilon_1 > 0$ establishes the resolution of the search, $h_o > 0$ defines the horizon of the search and, int(.) is the integer part of the argument. The upperbound $\varepsilon_1 < \pi$ is imposed to allow the reconstruction of $g(t)$ from its samples, as can be concluded from the Nyquist-Shannon Theorem applied in the worst case, i.e., the eigenvalues with the largest norm are complex and poorly damped, resulting in almost sinusoidal natural mode. On the other hand, the horizon should be large enough ($h_o \gg 1$) to allow the evaluation of the effect of the slow natural modes. Suggested values are $\varepsilon_1 = 0.1$ and $h_o = 10$, which have been rendering accurate numerical results with a small computational cost.

Trust region

To accelerate the execution of (46) in computers, the search can be restricted to the *trust region*

$$\mathcal{T} = \{k_i \in [0, \ldots, k_f - 1] \,:\, \|\hat{g}(k_i h)\| > \|\hat{g}(kh)\| \,, \forall k \in [k_i + 1, \ldots, k_f]\} \cup \{k_f\} , \tag{50}$$

which is justified by the fact that $\forall \gamma, h > 0$ and $\forall k \in \{k_j \in [0, \ldots, k_f] : k_j \notin \mathcal{T}\}$, we have that $\exists k_i > k$, $k_i \in \mathcal{T}$, such that $\|\hat{g}(kh)\| \exp\{\gamma k h\} < \|\hat{g}(k_i h)\| \exp\{\gamma k_i h\}$, which justifies the exclusion of the sample k from the *trust region*.

To illustrate the adavantage of restricting the search to $\mathcal{T}$, lets consider the computation of $\hat{C}_1(\gamma, h, k_f)$ for the filter with transfer function

$$G_2(s) = \frac{(s+7)(s-1+4j)(s-1-4j)}{(s+1)^3(s+0.5+10j)(s+0.5-10j)}\,, \tag{51}$$

with $h = 0.01\,s$ and $k_f = 400$. The number of samples in the *trust region* is 88, which is less than 22% of the total of 401 samples. Figure 2 shows the graphics of $|g_2(t)|$ where the *thrust region* segments are marked.

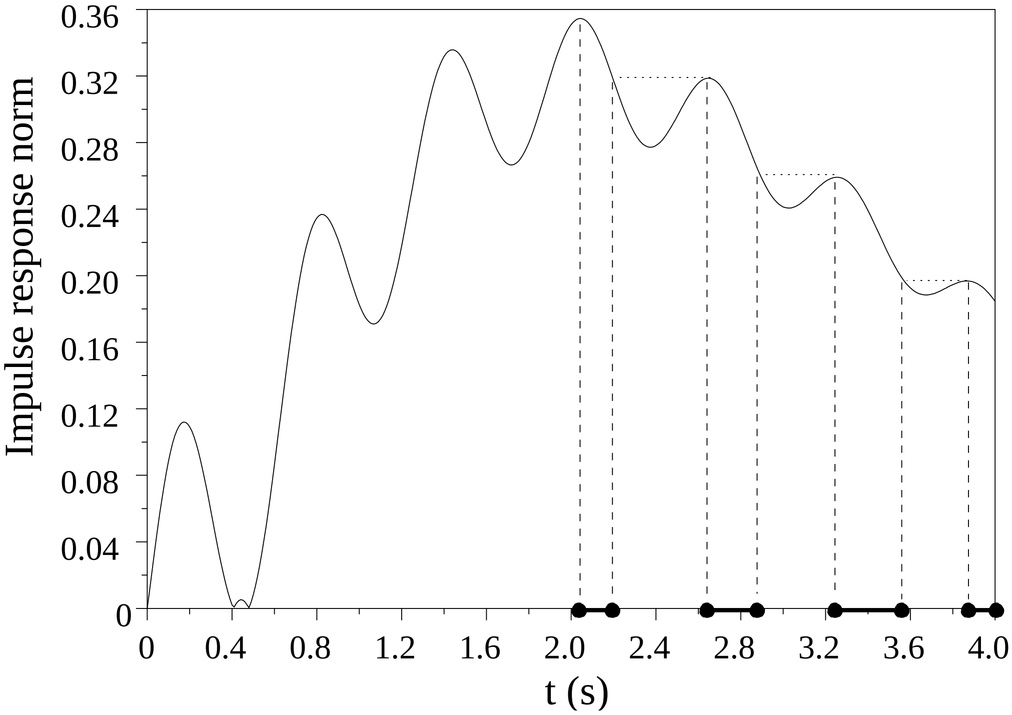

Figure 2: Norm of the impulse response of the filter $G_2(s)$. The *trust region* segments are marked in the time axis (bold).

The *thrust region* can be further streched taking advantage of previous computations of $\hat{C}_1$ for the same impulse response but for different values of γ. Once $\hat{C}_1(\gamma_{i-1}, h, k_f)$ has been computed for some value of γ subject to $\gamma_{i-1} > \gamma_i > 0$, the new end of the search $(k_{fi} \leq k_{fi-1} \leq k_f)$ can be chosen such that

$$\begin{aligned} \frac{\|\hat{g}(kh)\|}{\exp\{-\gamma_{i-1}kh\}} &< \hat{C}_1(\gamma_{i-1}, h, k_f) \quad \text{if } k < k_{fi}\,, \\ \frac{\|\hat{g}(kh)\|}{\exp\{-\gamma_{i-1}kh\}} &\leq \hat{C}_1(\gamma_{i-1}, h, k_f) \quad \text{if } k > k_{fi}\,, \end{aligned} \tag{52}$$

since it can be shown that $\|\hat{g}(kh)\| \exp\{\gamma_i kh\}$ has a maximum at k_i^* that preceds the maximum of $\|\hat{g}(kh)\| \exp\{\gamma_{i-1} kh\}$ at $k_{i-1}^* = k_{fi}$, i.e., $k_i^* \leq k_{i-1}^*$.

Analogously the begining of the *trust region* can be refined taking into account some previous computation of $\hat{C}_1(\gamma_{i-1}, h, k_f)$ with $\gamma_i > \gamma_{i-1} > 0$.

The stretching of the *trust region* is illustrated by the comparison of the norm of the impulse response $|g_2(t)|$ with two exponential functions $c_i \exp\{-\gamma_i t\}$ presented in Fig. 3, with $\gamma_1 = 0.499\, s^{-1}$, $\gamma_2 = 0.339\, s^{-1}$ and $c_i = \hat{C}_1(\gamma_i, 0.01, 400)$. As predicted the tangent points of $|g_2(t)|$ and the exponential "2" (at $t \approx 2.7\, s$ and $t \approx 3.3\, s$) preced the tangent point of $|g_2(t)|$ and the exponential "1" (at $t \approx 3.9\, s$), since $\gamma_2 < \gamma_1$.

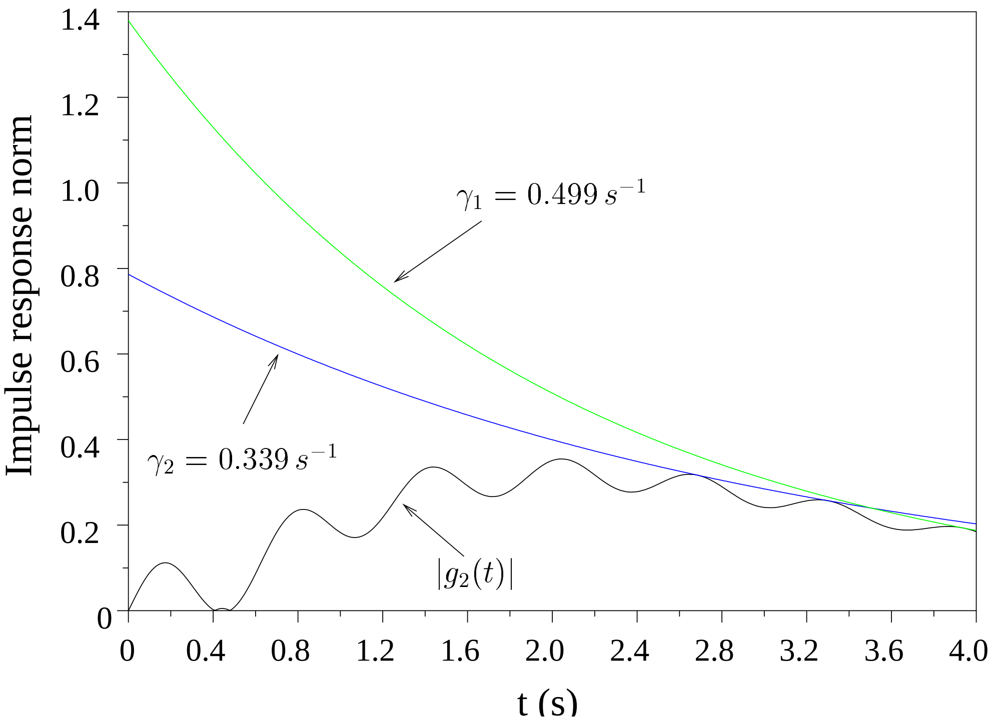

Figure 3: Norm of the impulse response of the filter $G_2(s)$ and the impulse responses of two FOAFs with distinct values of γ.

4.3.2 Minimization of the DC gain

The problem of minimizing the FOAF DC gain stated in Section 4.3 can be redefined as

$$\min_{\delta_l \leq \gamma \leq \gamma_0 - \delta_u} \hat{S}(\gamma, h, k_f)\,, \tag{53}$$

with

$$\hat{S}(\gamma, h, k_f) := \frac{\hat{C}_1(\gamma, h, k_f)}{\gamma}, \tag{54}$$

resulting in the optimal solution being at some point (c_1^*, γ^*) with $c_1^* = \hat{C}_1(\gamma^*, h, k_f)$. The nature of the DC gain function is illustrated by the plot of $\hat{S}(\gamma, 0.01, 400)$ obtained for the filter with transfer function $G_2(s)$ (Fig. 4) and the plot of the derivative $\partial\hat{S}(\gamma, 0.01, 400)/\partial\gamma$ computed numerically through a finite difference approximation (Fig. 5). The minimum $\hat{S}$ is obtained for $\gamma^* \approx 0.34\,(s^{-1})$. The derivative $\partial\hat{S}/\partial\gamma$ is discontinuous in three points as can be seen in Fig. 5, particullarly in the minimum, which precludes the application of derivative based methods to the optimization of $\hat{S}$.

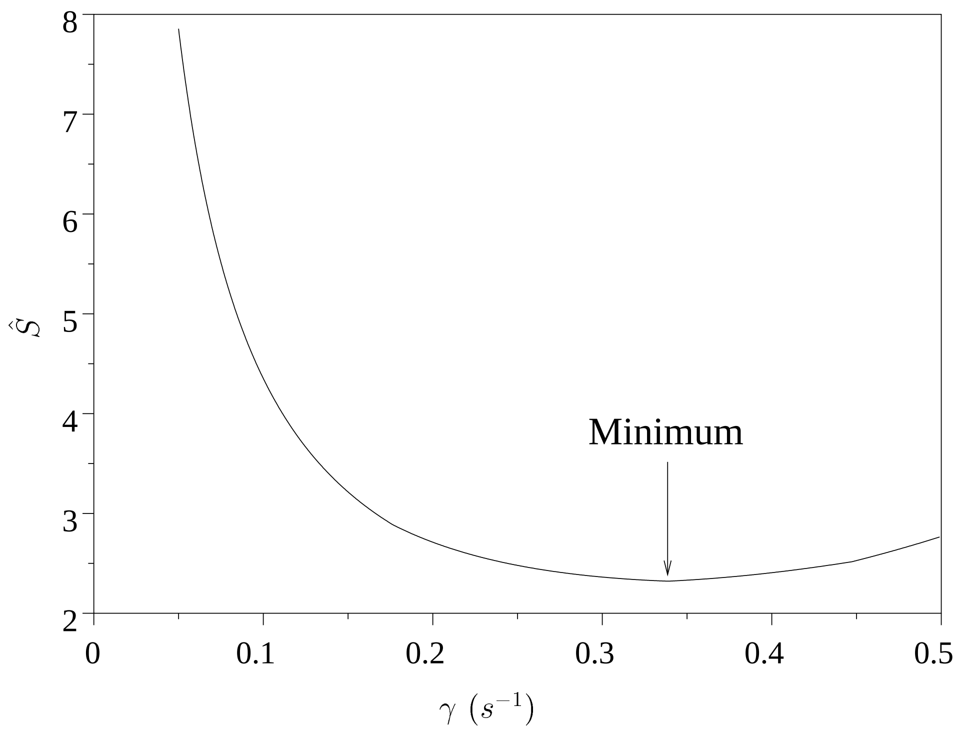

Figure 4: Plot of the DC gain of the FOAF designed for the filter $G_2(s)$ $(\hat{S}(\gamma, 0.01, 400))$ versus the design parameter γ.

Proposition 1 *Let $h > 0$, $k_f \in \mathbb{N}$ and $\gamma_0 - \delta_u \geq \delta_l > 0$ be arbitrary constants. Then, the function $\hat{S}(\gamma, h, k_f)$ defined in (54) is convex on the domain $\mathcal{D} := \{\gamma \in \mathbb{R}^+ : \delta_l \leq \gamma \leq \gamma_0 - \delta_u\}$.*

Proof: See appendix A.

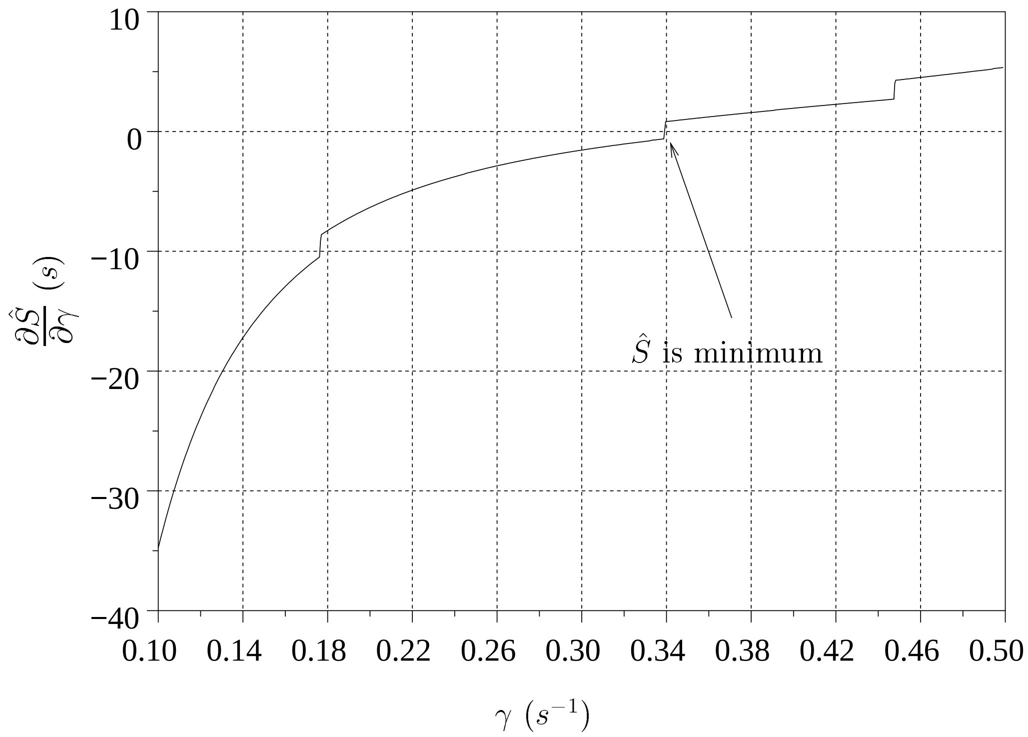

Figure 5: Plot of the derivative of the DC gain function $(\partial \hat{S}(\gamma, 0.01, 400)/\partial\gamma)$ versus the design parameter γ.

The convextity of $\hat{S}(\gamma, h, k_f)$ with respect to γ in the domain $[\delta_l, \gamma_0 - \delta_u]$ implies its unimodality, and then the *Golden Section Search* (e.g., (Van Loan 2000)) can be applied to the minimization (53). In this method, an interval $[\gamma_{1i}, \gamma_{4i}]$ that brackets γ^* is mainteined from step to step in addition to a pair of "interior" points γ_{2i} and γ_{3i} that satisfy $\gamma_{1i} < \gamma_{2i} < \gamma_{3i} < \gamma_{4i}$ and

$$\gamma_{2i} = \gamma_{1i} + r\,(\gamma_{4i} - \gamma_{1i})\,, \tag{55}$$
$$\gamma_{3i} = \gamma_{1i} + (1 - r)\,(\gamma_{4i} - \gamma_{1i})\,, \tag{56}$$

for some chosen factor $r \in (0, 0.5)$. The *Golden Section Search* applies the *Golden Ratio* $r_* = (1 + \sqrt{5})/2$ which generally gives a fast convergence of the algorithm. Since $r_* = (1 - r)^{-1}$ we have that $r = (3 - \sqrt{5})/2 \approx 0.3819660$.

If $\hat{S}(\gamma_{2i}, h, k_f) \leq \hat{S}(\gamma_{3i}, h, k_f)$, then due to unimodality it follows that $\hat{S}$ is increasing in $[\gamma_{3i}, \gamma_{4i}]$ and so $\gamma^* \in [\gamma_{1i}, \gamma_{3i}]$. On the other hand, if $\hat{S}(\gamma_{2i}, h, k_f) > \hat{S}(\gamma_{3i}, h, k_f)$ then a similar argument tells us that $\gamma^* \in [\gamma_{2i}, \gamma_{4i}]$. This suggests the algorithm described in Table 2.

Note that the counter variable i is not necessary and the indexed variables can be replaced by scalar variables. The stoping criteria $(\gamma_{3i} - \gamma_{1i} \leq \varepsilon_2\gamma_{1i})$ is chosen such that the

Table 2: DC gain minimization algorithm based on *Golden Section Search*.

```
begin
   γ11 = δl
   γ41 = γ0 − δu
   compute γ21 and γ31 through (55) and (56)
   i = 1
   while (γ3i − γ1i) > (ε2 γ1i) do
     if Ŝ(γ2i, h, kf) ≤ Ŝ(γ3i, h, kf) then
       γ1i+1 = γ2i
       γ2i+1 = γ3i
       γ4i+1 = γ4i
       compute γ3i+1 through (56)
     else
       γ4i+1 = γ3i
       γ3i+1 = γ2i
       γ1i+1 = γ1i
       compute γ2i+1 through (55)
     end
     i = i + 1
   end
   select Ŝ(γ̂*, h, kf) = min{Ŝ(γ1i, h, kf), Ŝ(γ2i, h, kf), Ŝ(γ3i, h, kf), Ŝ(γ41, h, kf)} and
   obtain the corresponding γ̂*
   ĉ* = γ̂* Ŝ(γ̂*, h, kf)
end
```

relative error on the estimate $\hat{\gamma}^*$, given by $|\hat{\gamma}^* - \gamma^*|/\gamma^*$, is kept within the tolerated margin ($\varepsilon_2 > 0$). Of course, the number of iterations (i) grows as the tolerated relative error is smaller and the initial search interval $[\delta_l, \gamma_0 - \delta_u]$ is wider.

An outstading property of the *Golden Section Search* algorithm is that the estimated $\hat{\gamma}^*$ always converges to the optimal γ^* as $i \to +\infty$, since $\hat{S}$ is *unimodal* across the initial interval (Van Loan 2000).

4.3.3 Optimal synthesis example

Optimal first order approximation filters are designed for the norm of the output signal of the filter with transfer function $G_1(s)$ (see (34)). The synthesis results obtained through the algorithms described above in this section for the three optimization objectives are presented on Table 3. The design parameters are: $\delta_l = \delta_u = 0.1\, s^{-1}$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 10^{-6}$ and $h_o = 10$. The impulse responses of each first order approximation filter and the norm of the impulse response of $G_1(s)$ ($|g_1(t)|$) are presented in Fig. 6.

Table 3: Synthesis results for the three optimization objetives applied to the filter $G_1(s)$.

	Optimal synthesis results		
Objective	γ (s^{-1})	c_1	DC gain
Minimal time constant	0.90	7.2	8.0
Minimal DC gain	0.69	4.9	7.1
Minimal peak value	0.10	2.7	26.9

Comparing these results with those obtained through Lyapunov synthesis presented on Table 1, we note that the DC gains and the peak values of the optimal designs are much lower than the Lyapunov results. An exception is that the DC gain for the minimal peak value design (26.9) is larger than the smallest DC gain obtained through the Lyapunov synthesis (19.6) for the cascade realization of $G_1(s)$. This is not surprising, since the DC gain for the minimal peak value design tends to infinity as the specified bound $\delta_l \to 0+$.

Due to the chosen values for the parameters ε_1 and h_o, 201 samples of the impulse response $\hat{g}_1(kh)$ are computed for the evaluation of $\hat{C}_1$. The corresponding sampling period is $h = 0.05\,s$. The time horizon $t_f = 10\,s$ seems to be overestimated since in Fig. 6 the norm of the impulse response is negligible for practical purposes after $t = 5\,s$, suggesting that the recommended value of the horizon parameter could be $h_o = 5$ in this example.

5 Conclusion

In this report the problem of generating an upperbound for the output signals of multi-input-multi-output linear time invariant filters (LTI) is revisited aiming at the design of sliding mode control systems.

Theorem 1 shows the existence of a first order approximation filter (FOAF) for the norm of the output signal of the LTI filter. Theorem 2 extends the applicability of the FOAF to filters with discontinuous input signals generated by sliding mode systems.

Optimization algorithms were developed to the design of the FOAF coefficients in order to meet optimality with respect of one of the three proposed performance criteria, namely: (1) minimum time constant; (2) minimum DC gain; or (3) minimum peak value of the impulse response. The proposed algorithms are trickly since the design of the FOAF may result in non-convex and non-differentiable optimization problems. The optimum design has a much better performance than is usually obtained through a Lyapunov approach.

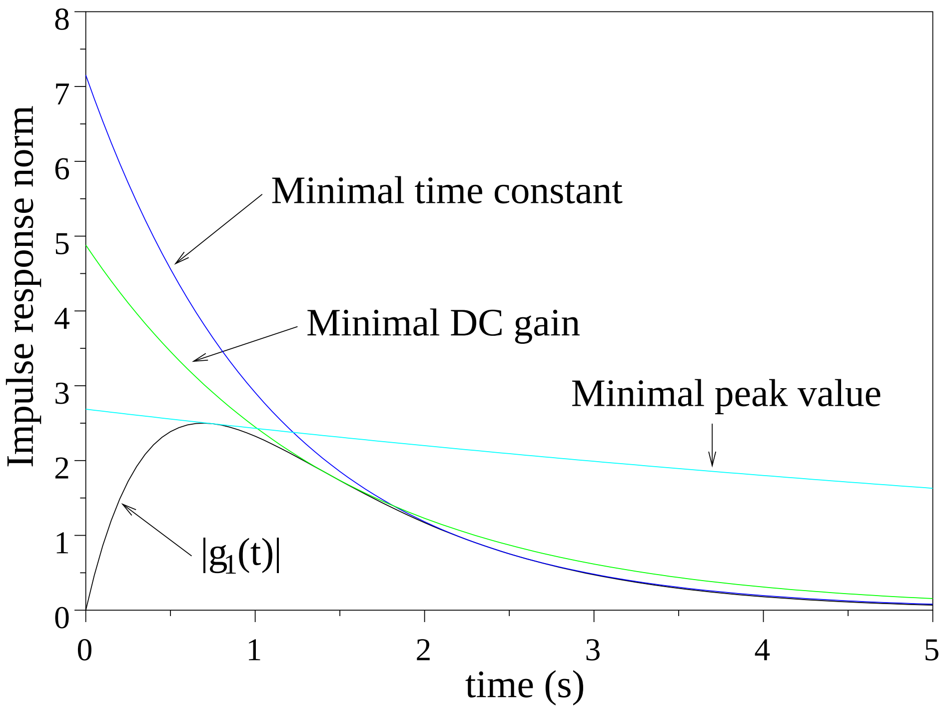

Figure 6: Impulse response of each optimal first order approximation filter and the norm of the impulse response of $G_1(s)$ ($|g_1(t)|$).

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Appendices

A Proof of Proposition 1

The initial step to prove this proposition is to rewrite (54) as

$$\hat{S}(\gamma, h, k_f) = \frac{\max_{k\in\mathcal{T}}\{\|\hat{g}(kh)\|\exp\{\gamma kh\}\}}{\gamma} = \max_{k\in\mathcal{T}}\{f(\gamma,h,k)\}\ , \tag{57}$$

where the $\mathcal{T} \subset \{0, 1, \ldots k_f\}$ is the *trust region* (see (50)) and

$$f(\gamma,h,k) := \|\hat{g}(kh)\|\frac{\exp\{\gamma kh\}}{\gamma}\,. \tag{58}$$

As an intermediate step we prove that if k and h are kept constant, then f is convex in the domain $\mathcal{D}$. For this purpose we use the second partial derivative

$$\frac{\partial^2 f}{\partial\gamma^2} = \|\hat{g}(kh)\|\frac{(\gamma kh)^2 - 2\gamma kh + 2}{\gamma^3}\exp\{\gamma kh\}\,, \tag{59}$$

which is continuous with respect to γ and positive provided that $\gamma > 0$. These characteristics imply the convextity of f over $\mathcal{D}$, as can be concluded from (Chandrupatla & Belegundu 1999, p. 25).

To complete the proof, we apply the definition of a convex function: $\hat{S}$ is a convex function on the convex domain $\mathcal{D}$ if for every pair γ_1 and γ_2 in $\mathcal{D}$ and $0 \leq \alpha \leq 1$, the following condition is satisfied (e.g., (Chandrupatla & Belegundu 1999, p. 25))

$$\hat{S}(\alpha\gamma_1 + (1-\alpha)\gamma_2, h, k_f) \leq \alpha\hat{S}(\gamma_1, h, k_f) + (1-\alpha)\hat{S}(\gamma_2, h, k_f) \,. \tag{60}$$

Considering (57), we can write that $\exists \bar{k} \in \mathcal{T}$ such that

$$\hat{S}(\alpha\gamma_1 + (1-\alpha)\gamma_2, h, k_f) = f(\alpha\gamma_1 + (1-\alpha)\gamma_2, h, \bar{k}) \,, \tag{61}$$

where $\gamma_1, \gamma_2 \in \mathcal{D}$ and $\alpha \in [0,1]$. On the other hand, since f is convex in $\mathcal{D}$, the following inequality is satisfied in the same domain

$$\begin{aligned} f(\alpha\gamma_1 + (1-\alpha)\gamma_2, h, k) &\leq \alpha f(\gamma_1, h, k) + (1-\alpha) f(\gamma_2, h, k) & (62) \\ &\leq \alpha \max_{k\in\mathcal{T}} \{f(\gamma_1, h, k)\} + (1-\alpha) \max_{k\in\mathcal{T}} \{f(\gamma_2, h, k)\} & (63) \\ &= \alpha\hat{S}(\gamma_1, h, k_f) + (1-\alpha)\hat{S}(\gamma_2, h, k_f) \,. & (64) \end{aligned}$$

$\forall k \in \mathcal{T}$, $\forall \gamma_1, \gamma_2 \in \mathcal{D}$ and $\forall \alpha \in [0,1]$. Substituting k by $\bar{k}$ in (62) and applying (61), we prove that inequality (60) is satisfied, thus proving that $\hat{S}$ is convex over the domain $\mathcal{D}$.

B Computer Algorithms for the Design of First Order Approximation Filters

This appendix includes the listings of two algorithms developed for Scilab 2.6. Scilab is a free software for numerical computing that can be downloaded from

`http://www-rocq.inria.fr/scilab` .

The function `first_order_filter` applies the optimization algorithms proposed in section 4.3. The function `first_order_lyapunov_filter` applies Lyapunov synthesis.

```
//-------------------------------------------------------------------------------
//
// Function to the synthesis of a first order nonlinear filter for approximation
// of the norm of the output signal of a linear multivariable filter. The method
// applies optimization with three distinct objectives: (i) minimize the filter
// DC gain, (ii) minimize the filter time constant and (iii) minimize the peak
// of the impulse response of the filter. This function also returns the
// Euclidian norm of the original filter impulse response.
//
//   Rio de Janeiro, May 13, 2002.
//
//   Author: Jose' Paulo V. S. da Cunha
//
//   Last review: June 04, 2002.
//
//-------------------------------------------------------------------------------
//
// Variables:
//
// A,B,C                  - Matrices of the linear filter realization
//                          (G(s) = C(sI-A)^-1 B).
// delta_u                - Design parameter that defines the upperbound
//                          for the feasible Gamma (gamma_max = gamma_0 - delta_u).
// delta_l                - Lower bound for the feasible Gamma.
// epsilon1               - Parameter in the range (0,pi). The error becomes
//                          larger as 'epsilon1' grows. Recommended value: 0.1.
// epsilon2               - Maximum allowed relative error on Gamma.
// ho                     - Impulse simulation horizon tf = ho/min(|spec(A)|).
//                          Recommended value: 10.
// c, Gamma               - Coefficients for the minimum DC gain filter
//                          (c/(s+Gamma)).
// c_gamma_max,gamma_max - Coefficients for the minimum time constant filter
//                          (c_gamma_max/(s+gamma_max)).
// c_delta_l              - Coefficient for the filter with minimum peak value
//                          of the impulse response (c_delta_l/(s+delta_l)).
// g_norm                 - Linear filter impulse response norm
//                          (||g(t)||=||Cexp(At)B||).
// t                      - Time vector corresponding to g_norm.
//
```

```
// Notes:
//
// 1) ||g(t) * u(t)|| <= c exp(-Gamma t) * ||u(t)||.
//
// 2) gamma_0 is the stability margin of matrix A.
//
// 3) delta_l<= Gamma <= gamma_max,  gamma_max = gamma_0 - delta_u.
//
//-----------------------------------------------------------------------------

function [c,Gamma,c_gamma_max,gamma_max,c_delta_l,t,g_norm]=
          first_order_filter(A,B,C,delta_u,delta_l,epsilon1,epsilon2,ho)

//
// Stops if some parameter is out of range:
//
if (delta_u<=0) | (delta_l<=0) | (epsilon1<=0) | (epsilon1>=%pi)
   | (epsilon2<=0) | (ho<=0) then
   error('Cannot design FOAF: some input parameter out of range.');
end;

//
// Eigenvalues of the filter realization:
//
eigenvalues = spec(A);

//
// Stability margin:
//
gamma_0 = -max(real(eigenvalues));

//
// Stops if the filter is unstable:
//
if gamma_0 <=0 then
   error('Cannot design FOAF: unstable filter.');
end;

//
// Step size (h) and time range (kf) selection for filter impulse
// response simulation:
//
abs_eigenvalues = abs(eigenvalues);

h = epsilon1/max(abs_eigenvalues);

kf = int(0.5+ho/(min(abs_eigenvalues)*h));

kf1 = kf+1;

//
```

```
// Upperbound for gamma:
//
gamma_max = gamma_0 - delta_u;

//
// Stops if the design parameters delta_u and delta_l are too large:
//
if gamma_max < delta_l then
   error('Cannot design FOAF: (delta_u+delta_l) is too large.');
end;

//
// Impulse response computation:
//
PHI = expm(A*h);

X = B;

t = zeros(kf1);

g_norm = zeros(kf1);

g_norm(1) = norm(C*B,2);

k = 1;

while k<=kf do
   k1 = k+1;
   X = PHI*X;
   t(k1) = h*k;
   g_norm(k1) = norm(C*X,2);
   k = k1;
end;

// Find the thrust region (T):

T = zeros(kf1);

T(1) = kf1;

kbegin = 1;

g_norm_peak = g_norm(kf1);

k = kf;

while k>=1 do
  if g_norm(k)>g_norm_peak then
    kbegin = kbegin+1;
    T(kbegin) = k;
    g_norm_peak = g_norm(k);
```

```
  end;
  k = k-1;
end;

//
// Find c_gamma_max = C1(gamma_max):
//
gamma_max_h = gamma_max*h;

exp_gamma_h = exp(-gamma_max_h);

exp_gamma_k = exp(-gamma_max_h*(T(kbegin)-1));

C1 = g_norm(T(kbegin))/exp_gamma_k;

k = kbegin;

kfinal = 1;

while k>1 do
   k1 = k-1;
   if T(k1)==(T(k)+1) then
     exp_gamma_k = exp_gamma_k*exp_gamma_h;
   else
     exp_gamma_k = exp(-gamma_max_h*(T(k1)-1));
   end;
   relation = g_norm(T(k1))/exp_gamma_k;
//
// Find the peak value of the relation (norm/exp) and the first sample
// of its occurence:
//
   if relation>C1 then
     C1 = relation;
     kfinal = k1;
   end;
   k = k1;
end;

c_gamma_max = C1;

//
// Find c_delta_l = C1(delta_l):
//
delta_l_h = delta_l*h;

exp_gamma_h = exp(-delta_l_h);

exp_gamma_k = exp(-delta_l_h*(T(kbegin)-1));

C1 = g_norm(T(kbegin))/exp_gamma_k;
```

```
k = kbegin;

while k>kfinal do
   k1 = k-1;
   if T(k1)==(T(k)+1) then
     exp_gamma_k = exp_gamma_k*exp_gamma_h;
   else
     exp_gamma_k = exp(-delta_l_h*(T(k1)-1));
   end;
   relation = g_norm(T(k1))/exp_gamma_k;
//
// Find the peak value of the relation (norm/exp) and the last sample
// of its occurence:
//
   if relation>=C1 then
     C1 = relation;
     kbegin = k1;
   end;
   k = k1;
end;

c_delta_l = C1;

//
// Computation of c and Gamma through Golden Search:
//

// Golden ratio (r=(3-sqrt(5))/2, r1=1-r):
//
r = 0.3819660;

r1 = 0.6180340;

// Initial points:
//
gamma1 = delta_l;

gamma4 = gamma_max;

gamma2 = gamma1*r1+gamma4*r;

gamma3 = gamma1*r+gamma4*r1;

// Peak values and corresponding areas:

c_gamma1 = c_delta_l;

c_gamma4 = c_gamma_max;

S1 = c_gamma1/gamma1;
```

```
S4 = c_gamma4/gamma4;

//
// Find c_gamma2 = C1(gamma2):
//
gamma2_h = gamma2*h;

exp_gamma_h = exp(-gamma2_h);

exp_gamma_k = exp(-gamma2_h*(T(kbegin)-1));

C1 = g_norm(T(kbegin))/exp_gamma_k;

k = kbegin;

k2 = kbegin;

while k>kfinal do
   k1 = k-1;
   if T(k1)==(T(k)+1) then
     exp_gamma_k = exp_gamma_k*exp_gamma_h;
   else
     exp_gamma_k = exp(-gamma2_h*(T(k1)-1));
   end;
   relation = g_norm(T(k1))/exp_gamma_k;
//
// Find the peak value of the relation (norm/exp) and the last sample
// of its occurence:
//
   if relation>=C1 then
     C1 = relation;
     k2 = k1;
   end;
   k = k1;
end;

c_gamma2 = C1;

S2 = c_gamma2/gamma2;

//
// Find c_gamma3 = C1(gamma3):
//

gamma3_h = gamma3*h;

exp_gamma_h = exp(-gamma3_h);

exp_gamma_k = exp(-gamma3_h*(T(k2)-1));

C1 = g_norm(T(k2))/exp_gamma_k;
```

```
k = k2;

k3 = kfinal;

while k>kfinal do
   k1 = k-1;
   if T(k1)==(T(k)+1) then
     exp_gamma_k = exp_gamma_k*exp_gamma_h;
   else
     exp_gamma_k = exp(-gamma3_h*(T(k1)-1));
   end;
   relation = g_norm(T(k1))/exp_gamma_k;
//
// Find the peak value of the relation (norm/exp) and the first sample
// of its occurence:
//
   if relation>C1 then
     C1 = relation;
     k3 = k1;
   end;
   k = k1;
end;

c_gamma3 = C1;

S3 = c_gamma3/gamma3;

//
// Golden search loop:
//
while (gamma3-gamma1)>(epsilon2*gamma1) do
  if S2>=S3 then
    gamma1 = gamma2;
    gamma2 = gamma3;
    gamma3 = gamma1*r+gamma4*r1;
    c_gamma1 = c_gamma2;
    c_gamma2 = c_gamma3;
    S1 = S2;
    S2 = S3;
    kbegin = k2;
//
// Find c_gamma3 = C1(gamma3):
//
    gamma3_h = gamma3*h;
    exp_gamma_h = exp(-gamma3_h);
    exp_gamma_k = exp(-gamma3_h*(T(kbegin)-1));
    C1 = g_norm(T(kbegin))/exp_gamma_k;
    k = kbegin;
    k3 = kfinal;
    while k>kfinal do
```

```
       k1 = k-1;
       if T(k1)==(T(k)+1) then
         exp_gamma_k = exp_gamma_k*exp_gamma_h;
       else
         exp_gamma_k = exp(-gamma3_h*(T(k1)-1));
       end;
       relation = g_norm(T(k1))/exp_gamma_k;
//
// Find the peak value of the relation (norm/exp) and the first sample
// of its occurence:
//
       if relation>C1 then
         C1 = relation;
         k3 = k1;
       end;
       k = k1;
    end;
    c_gamma3 = C1;
    S3 = c_gamma3/gamma3;
  else
    gamma4 = gamma3;
    gamma3 = gamma2;
    gamma2 = gamma1*r1+gamma4*r;
    c_gamma4 = c_gamma3;
    c_gamma3 = c_gamma2;
    S4 = S3;
    S3 = S2;
    kfinal = k3;
//
// Find c_gamma2 = C1(gamma2):
//
    gamma2_h = gamma2*h;
    exp_gamma_h = exp(-gamma2_h);
    exp_gamma_k = exp(-gamma2_h*(T(kbegin)-1));
    C1 = g_norm(T(kbegin))/exp_gamma_k;
    k = kbegin;
    k2 = kbegin;
    while k>kfinal do
       k1 = k-1;
       if T(k1)==(T(k)+1) then
         exp_gamma_k = exp_gamma_k*exp_gamma_h;
       else
         exp_gamma_k = exp(-gamma2_h*(T(k1)-1));
       end;
       relation = g_norm(T(k1))/exp_gamma_k;
//
// Find the peak value of the relation (norm/exp) and the last sample
// of its occurence:
//
       if relation>=C1 then
         C1 = relation;
```

```
         k2 = k1;
       end;
       k = k1;
    end;
    c_gamma2 = C1;
    S2 = c_gamma2/gamma2;
  end;
end;

// Selects the sub-optimum with minimum DC gain (S):

S = S4;
c = c_gamma4;
Gamma = gamma4;

if S3 < S then
  S = S3;
  c = c_gamma3;
  Gamma = gamma3;
end;

if S2 < S then
  S = S2;
  c = c_gamma2;
  Gamma = gamma2;
end;

if S1 < S then
  c = c_gamma1;
  Gamma = gamma1;
end;

endfunction

//--------------------------------------------------------------------------------
//
// Function to the synthesis of a first order nonlinear filter for approximation
// of the norm of the output signal of a linear multivariable filter. The method
// applies Lyapunov synthesis and may give conservative results.
//
//   Rio de Janeiro, May 14, 2002.
//
//   Author: Jose' Paulo V. S. da Cunha
//
//   Last review: May 14, 2002.
//
//--------------------------------------------------------------------------------
//
// Variables:
//
// A,B,C                 - Matrices of the linear filter realization
```

```
//                          (G(s)=C(sI-A)^-1 B).
// c1, Gamma              - Coefficients for the filter (c1/(s+Gamma)).
// c2_                    - Coefficient for the transient term of the filter.
//
// Note: ||y(t)|| <= c1 exp(-Gamma t) * ||u(t)|| + c2 exp(-Gamma t) ||x(0)||.
//
//--------------------------------------------------------------------------------

function [c1,c2,Gamma]=first_order_lyapunov_filter(A,B,C)

// Solve the Lyapunov equation A'P+PA=-2I:

n = size(A,'c');

P = lyap(A,-2*eye(n,n),'c');

// Compute eigenvalues of P:

eigenvalues = spec(P);

lambda_min = min(eigenvalues);

lambda_max = max(eigenvalues);

// Compute the first order approximation filter pole:

Gamma = 1/lambda_max;

// Compute the first order approximation filter coefficient:

C_norm = norm(C,2);

c1 = C_norm*norm(P*B,2)/lambda_min;

// Compute the transient term coefficient:

c2 = C_norm*sqrt(lambda_max/lambda_min);

endfunction
```