Unit Vector Control of Uncertain Multivariable Systems

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Abstract— **This report presents several theorems on unit vector control in view of its application to model-reference sliding mode control of multivariable systems. A remarkable feature of the proposed method is that it requires that th high frequency gain matrix of the open loop system be Hurwitz. This relaxes the requirement of positive definiteness of the high frequency gain of usual methods.**

Keywords— **sliding mode, unit vector control, model-reference control, multivariable control.**

Preliminaries

A. Norms and notations

The following notation and basic concepts are employed in this chapter.

- The maximum and the minimum singular values of a matrix A are denoted as $\sigma_{max}(A)$ and $\sigma_{min}(A)$, respectively.
- The maximum and minimum eigenvalues of a symmetric matrix P are denoted as $\lambda_{max}(P)$ and $\lambda_{min}(P)$, respectively.
- $\|x\|$ denotes the Euclidean norm of a vector x and $\|A\| = \sigma_{max}(A)$ denotes the corresponding induced norm of a matrix A.
- The $\mathcal{L}_{\infty e}$ *norm* of the signal $x(t) \in \mathbb{R}^n$ is defined as [1]

$$\|x_t\|_\infty := \sup_{0\leq\tau\leq t} \|x(\tau)\| \ . \tag{1}$$

- Mixed time-domain and Laplace transform domain (operator) representations will be adopted. However, to give a precise meaning for such representations, the following notation and concepts are adopted. The output y of a linear time-invariant system with transfer function $H(s)$ and input u is given by $H(s)u$. Pure convolution operations $h(t) * u(t)$, $h(t)$ being the impulse response from $H(s)$, will be eventually written, for simplicity, as $H(s) * u$. Consider the realization $\dot{x} = Ax + Bu$, $y = Cx + Du$, of $H(s)$. Then,

$$y(t) = H(s)u(t) = h(t) * u(t) + C\exp(At)x(0) \,, \tag{2}$$

where the exponential term is the homogeneous response of the system ($u(t) \equiv 0$).

- The symbol s denotes either the complex variable of Laplace transforms or the differential operator $\frac{d}{dt}$ in time-domain expressions.

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- The norm of an operator $H(s)$ is defined as

$$\|H(s)\| := \|h(t)\|_1 = \int_0^{+\infty} \|h(\tau)\| d\tau \ . \tag{3}$$

The following inequality holds for $y(t) = h(t) * u(t)$ ([1], p. 80)

$$\|y_t\|_\infty \leq \|h(t)\|_1 \ \|u_t\|_\infty \, . \tag{4}$$

- A *stability margin* δ of a Hurwitz polynomial $p(\lambda)$ is defined as

$$0 < \delta \leq \min_i Re(-\lambda_i) \,, \tag{5}$$

where λ_i are the roots of $p(\lambda)$. Similarly, for a Hurwitz matrix A or a Hurwitz transfer function $G(s)$, a stability margin is defined with λ_i being the eigenvalues of A or the poles of $G(s)$.

- We assume $t \in \mathbb{R}^+$ so that $\forall t$ means $\forall t \geq 0$.
- We use "LI" to denote *locally integrable* in the sense of Lebesgue and omit the term *"almost everywhere"* since its need is believed obvious where necessary.
- We denote by $\pi(t)$ any exponentially decreasing signal, i.e., $\|\pi(t)\| \leq Re^{-\lambda t}$, $\forall t$, for some unknown positive scalars R and λ.

B. Discontinuous Differential Equations

- Since we shall deal with discontinuous systems, it is necessary to define precisely the meaning of a solution for such systems. Here, Filippov's definition is assumed [2]. Note that the control signal u is not necessarily a function of t in the usual sense when sliding modes take place. In order to avoid clutter, we will denote by $u(t)$ the locally integrable functions which are equivalent to u, in the sense of *equivalent control*, along any given Filippov solution $z(t)$ of the closed-loop system. It should be stressed that $z(t)$ is, *by definition*, absolutely continuous. This definition is motivated by the adequate representation of the behavior of physical systems when the actual switching mechanism tends to an ideal switching device which corresponds to the given discontinuous differential equation [3, pp. 94–99] [4]. Then, along any such solution, u can be replaced by $u(t)$ in the right-hand side of the governing differential equations. Although the equivalent control $u(t) = u_{eq}(t)$ is not directly available, for affine systems filtering u with any strictly proper filter $G(s)$ gives $G(s)u = G(s)u(t) = G(s)u_{eq}(t)$.
- The *extended equivalent control* is defined as an equivalent control which applies for a complete system motion,

i.e., on and outside the sliding mode surface $\sigma(x(t),t)=0$. Consider the class of nonlinear systems described by

$$\dot{x} = f(x,t) + B(x,t)u\,, \tag{6}$$

where $f(\cdot)$ and $B(\cdot)$ are smooth vector fields. Let $x(t)$ be a solution of this system, for $t \in [0,T)$. Then, the extended equivalent control is a locally integrable function, defined almost everywhere in the interval $[0,T)$, and is given by $u_{eq} = -[GB(x(t),t)]^{-1}[Gf(x(t),t) + \frac{d}{dt}\sigma(x(t),t)]$, where $G = \frac{\partial}{\partial x}\sigma(x(t),t)$. The above expression is well defined since the solution $x(t)$ is absolutely continuous by definition and, thus, has derivatives almost everywhere.

I. Introduction

In this paper we consider the unit vector control law originally proposed by Gutman and Leitmann [5], [6], [7]

$$u = -\rho(x,t)\frac{v(x)}{\|v(x)\|}\,, \tag{7}$$

where x is the state vector, $v(x)$ is a vector function of the (partial) state of the system (e.g., the output) and $\rho(x,t) \geq 0$, $\forall x$, $\forall t$. We refer to $\rho(\cdot)$ as the unit vector modulation function, which is designed to induce a sliding mode on the manifold $v(x)=0$. We will henceforth assume that $u=0$ if $v(x)=0$ only to have a complete definition of the control law.

A. Basic Lemmas

Some lemmas regarding the application of the unit vector control into the MRAC framework are now introduced. These lemmas generalize their SISO counterparts found in [8] and are instrumental for the controller synthesis and stability analysis.

Proposition 1: Consider the MIMO system

$$\dot{\varepsilon}(t) = A\varepsilon(t) + K[u + d(t) + \pi(t)]\,, \tag{8}$$

where $A, K \in \mathbb{R}^{m\times m}$, $d(t)$ and $\pi(t)$ are LI. Assume that $-K$ is Hurwitz. If

$$u = -\rho(\varepsilon,t)\frac{\varepsilon}{\|\varepsilon\|}\,, \tag{9}$$

$$\rho(\varepsilon,t) \geq \delta + c_\varepsilon\|\varepsilon(t)\| + (1+c_d)\|d(t)\|\,, \tag{10}$$

where ρ is LI, $c_\varepsilon \geq 0$ and $c_d \geq 0$ are appropriate constants, and $\delta \geq 0$ is an arbitrary constant, then, for the closed loop system (8)–(10), the inequality

$$\|\varepsilon(t)\| \leq (c_1\|\varepsilon(0)\| + c_2 R)\, e^{-\lambda_1 t} \tag{11}$$

holds $\forall t$ for some positive constants c_1, c_2 and λ_1. Therefore, the system is globally exponentially stable when $\pi(t) \equiv 0$. Moreover, if $\delta > 0$, then the sliding mode at the point $\varepsilon = 0$ is reached after some finite time $t_s \geq 0$.
Proof: see Appendix B-A.

Proposition 2: Consider the system

$$\dot{x} = K\frac{x}{\|x\|}\,, \tag{12}$$

where $x \in \mathbb{R}^m$, $m \geq 1$ and $K \in \mathbb{R}^{m\times m}$. The origin $x=0$ of of the system (24) is globally asymptotically stable if and only if K is Hurwitz.
Proof: The proof follows directly from Theorem 2 [9, Theorem 1].

Lemma 1: Consider the MIMO system

$$\varepsilon(t) = M(s)[u + d(t) + \pi(t)]\,, \tag{13}$$

where $M(s)$ is a minimum phase $(m\times m)$ transfer function matrix with uniform vector relative degree one and high frequency gain matrix K, $-K$ being Hurwitz, and $d(t)$ and $\pi(t)$ are LI. If

$$u = -\rho(\varepsilon,t)\frac{\varepsilon}{\|\varepsilon\|}\,, \tag{14}$$

$$\rho(\varepsilon,t) \geq \delta + c_\varepsilon\|\varepsilon\| + (1+c_d)\|d(t)\| + c_f\varepsilon_f\,, \tag{15}$$

where ρ is LI, $\delta \geq 0$ is an arbitrary constant, c_f, γ_0, c_ε, c_d are appropriate nonnegative constants, and ε_f is generated by the filter

$$\dot{\varepsilon}_f = -\gamma_0\varepsilon_f + \gamma_0\|\varepsilon\|\,, \tag{16}$$

then, the inequality

$$\|\varepsilon(t)\| \text{ and } \|x_{ea}(t)\| \leq [c_1\|x_{ea}(0)\| + c_2 R]\ e^{-\lambda_1 t} \tag{17}$$

holds $\forall t \geq 0$ for some positive constants c_1, c_2, λ_1, where $x_{ea}^T = [x_e^T\ \varepsilon_f]$ is the complete state of the closed loop system and x_e is the state of any stabilizable and detectable realization of (13) (possibly nonminimal). Moreover, if $\delta > 0$, then $\varepsilon(t)$ becomes identically zero after some finite time $t_s \geq 0$.
Proof: see Appendix B-B.

A more specialized version of Lemma 1 is the following corollary.

Corollary 1 (Lemma 1) Suppose, in Lemma 1, that $M(s) = \mathrm{diag}\{1/(s+\alpha_i)\}K$ with $-K$ Hurwitz. Then all the properties of Lemma 1 hold for

$$\rho(\varepsilon,t) \geq \delta + c_\varepsilon\|\varepsilon\| + (1+c_d)\|d(t)\|\,,\quad \forall t\,. \tag{18}$$

Moreover, if $\alpha_i = \alpha > 0$, $(\forall i)$, then $c_\varepsilon = 0$.
Proof: see Appendix B-C.

Lemma 2: Consider the MIMO system

$$\begin{aligned} \dot{\hat{\varepsilon}}(t) &= -\alpha\hat{\varepsilon}(t) + K[u + d(t)]\,, \\ \varepsilon(t) &= \hat{\varepsilon}(t) + \pi(t) + \beta(t)\,, \end{aligned} \tag{19}$$

which has input-output relationship given by

$$\varepsilon(t) = L^{-1}(s)K[u + d(t)] + \pi(t) + \beta(t)\,, \tag{20}$$

where $L(s) = (s+\alpha)I$, I is the $m\times m$ identity matrix, $\hat{\varepsilon}, \varepsilon, u, d \in \mathbb{R}^m$, $\alpha > 0$, $-K \in \mathbb{R}^{m\times m}$ is Hurwitz, $d(t)$ is LI, and $\pi(t)$ and $\beta(t)$ are absolutely continuous, $(\forall t)$. If $u = -\rho(t)\frac{\varepsilon}{\|\varepsilon\|}$, where ρ is LI and $\rho(t) \geq (1+\hat{c}_d)\|d(t)\|$, $\forall t$, for some appropriate $\hat{c}_d \geq 0$, then, $\exists c_1, c_2 > 0$ such that the signals $\varepsilon(t)$ and $\hat{\varepsilon}(t)$ are bounded by

$$\begin{aligned} &\|\varepsilon(t)\| \text{ and } \|\hat{\varepsilon}(t)\| \leq \\ &\leq c_1\|\hat{\varepsilon}(0)\|\ e^{-\alpha t} + c_2\left[Re^{-\min(\alpha,\lambda)t} + \|\beta_t\|_\infty\right]\,. \end{aligned} \tag{21}$$

Proof: see Appendix B-D.

II. Unit Vector Versus Variable Structure Control

Traditional VSC systems are based on the sign function ($x \in \mathbb{R}^m$)

$$\operatorname{sgn}(x) := \begin{bmatrix} \operatorname{sgn}(x_1) \\ \vdots \\ \operatorname{sgn}(x_m) \end{bmatrix}, \ \operatorname{sgn}(x_i) := \begin{cases} 1 & \text{if } x_i > 0, \\ 0 & \text{if } x_i = 0, \\ -1 & \text{if } x_i < 0. \end{cases} \quad (22)$$

To compare the stability properties of VSC and UVC, let us consider the VSC system given by

$$\dot{x} = K\operatorname{sgn}(x)\,, \quad K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, \quad x \in \mathbb{R}^2\,. \quad (23)$$

The following Theorem was proved in [10].

Theorem 1: Consider the discontinuous system (23). Then $x(t) = 0$ is a globally asymptotically stable solution if and only if one of the following conditions holds:
(1) $k_{12}k_{21} = 0$, $k_{11} < 0$ and $k_{22} < 0$.
(2) $k_{12}k_{21} \neq 0$, $\frac{k_{11}}{|k_{12}|} + \frac{k_{22}}{|k_{21}|} < 0$, as well as $\det(K) > 0$.

Equivalent conditions are also given in [3, pp. 256–258]. Unfortunately, no necessary and sufficient conditions are known for systems of dimension greater than two. In contrast, the following Theorem for UVC systems of arbitrary dimension was proved in [9].

Theorem 2: Consider the system

$$\dot{x} = K(x)\frac{x}{\|x\|}\,, \quad (24)$$

where $x \in \mathbb{R}^m$, $m \geq 1$, $K : \mathbb{R}^m \to \mathbb{R}^{m \times m}$, and $\det(K(x)) \neq 0$, $\forall x$. The origin of the state-space of system (24), with bounded $K(x)$ and its derivatives, is stable (asymptotically stable, unstable) if and only if the system

$$\dot{z} = K(z)z \quad (25)$$

is stable (asymptotically stable, unstable).

In particular, if $K(x)$ is a constant matrix, we conclude that the origin of the UVC system (24) is globally asymptotically stable if and only if K is Hurwitz. It is straightforward to find matrices K which result in asymptotic stability with both control laws, e.g., $K = -I$. However there exist matrices which result in stable closed-loop systems with one control law only, for example the matrix

$$K_1 \quad = \quad \begin{bmatrix} -1.5 & -2 \\ 1 & 1 \end{bmatrix}, \quad (26)$$

results in stable UVC only, and the matrix

$$K_2 \quad = \quad \begin{bmatrix} -0.5 & 1 \\ -5 & 1 \end{bmatrix}, \quad (27)$$

results in stable VSC only. To compare the performance of the control laws, simulation results with UVC (24) and VSC (23) for $K = K_1$ and $K = K_2$ are presented in Figs. 1 and 2. In these simulations, the initial condition is $x(0) = [1\ 2]^T$.

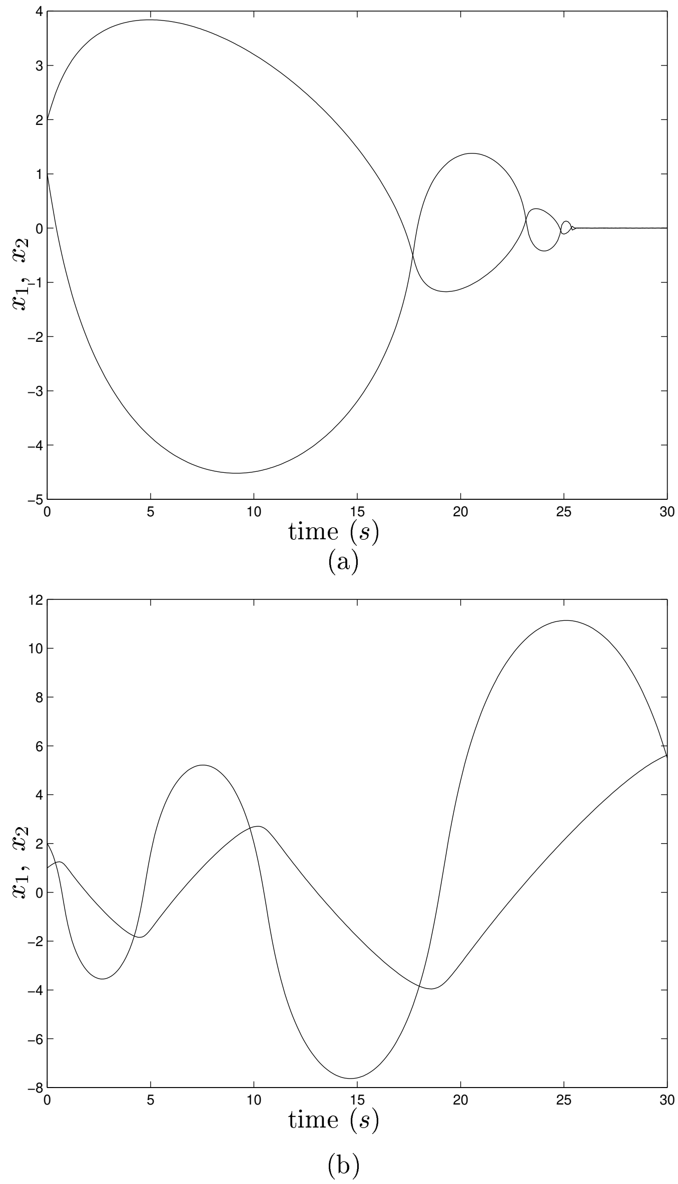

Fig. 1. Simulation of unit vector control system (24) with (a) $K = K_1$ and (b) $K = K_2$. The closed loop system is unstable with K_2

Another clear difference between VSC and UVC is that sliding modes in the former may take place in any individual switching surfaces before reaching their intersection $x = 0$, whereas, for UVC, the sliding mode occurs only at the origin. This is observed in the phase portraits of Fig. 3. For the VSC case it is shown that (Fig. 3.b) $x_1 = 0$ and $x_2 = 0$ are sliding surfaces, as well as their intersection $x = 0$. However, for the UVC (Fig. 3.a) the sliding mode is restricted to $x = 0$.

Appendix

I. Complementary Lemmas

Lemma 3: Let $r(t)$ be an absolutely continuous scalar function. Suppose $r(t)$ is nonnegative and while $r > 0$ it satisfies $\dot{r} \leq -\delta - \gamma r + Re^{-\lambda t}$, where $\delta, \gamma, \lambda, R$ are nonnegative constants. Then, one can conclude that: **(a)** $r(t)$ is

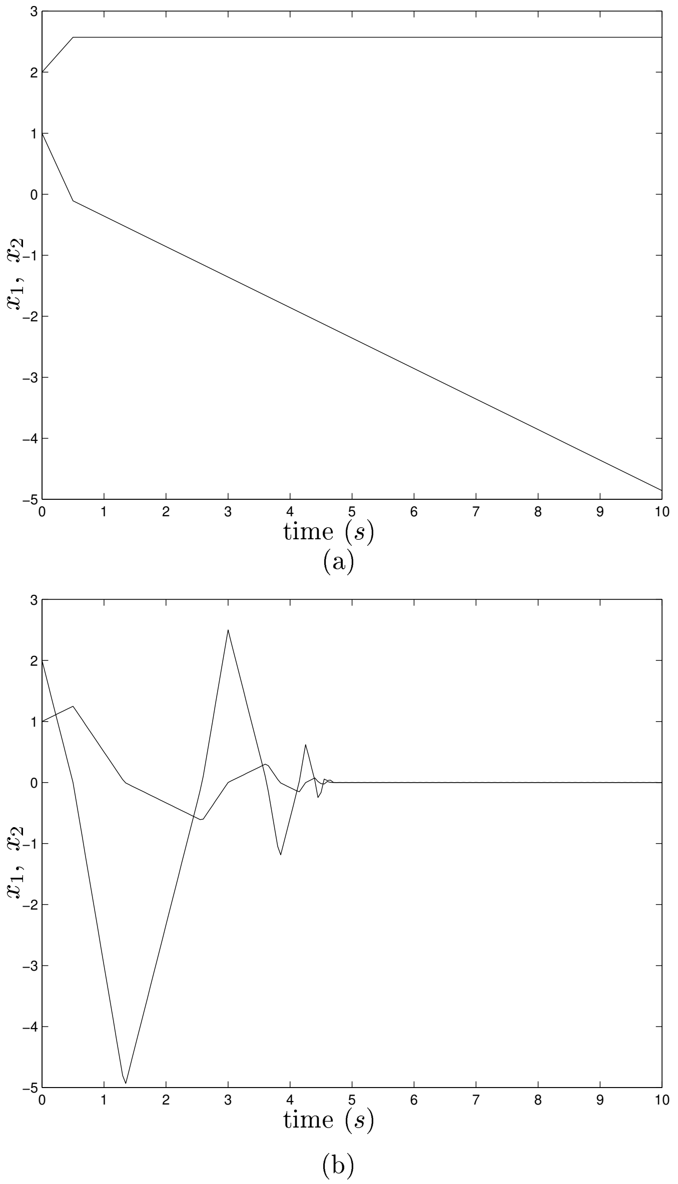

Fig. 2. Simulation of sign function control system (23) with (a) $K = K_1$ and (b) $K = K_2$. The closed loop system is unstable with K_1

bounded by

$$r(t) \leq [r(0) + cR]e^{-\min(\lambda,\gamma)t} \ (\forall t \geq 0), \tag{28}$$

where c is an appropriate positive constant; **(b)** if $\delta > 0$ then there exists $t_s < +\infty$ such that $r(t) \equiv 0$ for all $t \geq t_s$.
Proof: According to the Comparison Theorem [2], an upper bound $\bar{r}(t)$ of $r(t)$, is given by the solution of:

$$\dot{\bar{r}} = -\gamma\bar{r} + Re^{-\lambda t}, \qquad \bar{r}(0) = r(0).$$

Hence, $r(t) \leq [r(0) + cR]e^{-\min(\gamma,\lambda)t}$, which proves the first part of the Lemma. Now, if $\delta > 0$, the comparison equation is

$$\dot{\bar{r}} = -\delta - \gamma\bar{r} + Re^{-\lambda t}, \qquad \bar{r}(0) = r(0), \tag{29}$$

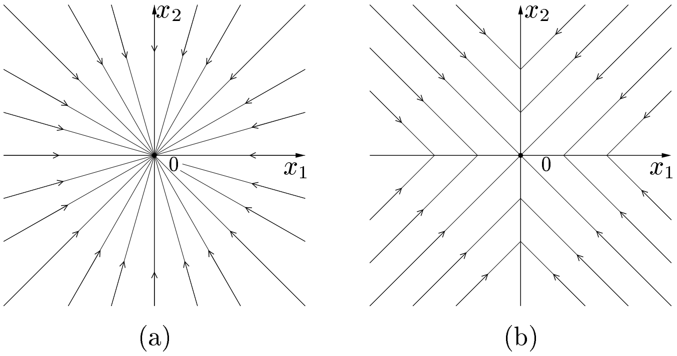

Fig. 3. Phase portraits of (a) unit vector control and (b) sign function control. The HFG matrix is $K = -I$

which implies that while $r(t) > 0$ one has

$$r(t) \leq [r(0) + \delta/\gamma + cR]e^{-\min(\gamma,\lambda)t} - \delta/\gamma.$$

Defining $t_s = [\min(\lambda, \gamma)]^{-1}\ln\{\gamma(r(0)+\delta/\gamma+cR)/\delta\} < +\infty$, one notes that the right hand side of the latter inequality is negative $\forall t > t_s$. Thus, since $r(t)$ continuous and $r(t) \geq 0$ one concludes that $r(t)$ becomes identically zero for $t \geq t_s$. ■

Lemma 4: Consider the system

$$z = W(s)d, \qquad z \in \mathbb{R}^m, \quad d \in \mathbb{R}^p, \tag{30}$$

where $W(s)$ is a stable and strictly proper $m \times p$ transfer matrix. Let γ_0 be the stability margin of $W(s)$, i.e., $0 < \gamma_0 < \min_j|Re(p_j)|$, where p_j are the poles of $W(s)$. Let $\bar{d}(t)$ be an instantaneous upper bound of $d(t)$, i.e., $\|d(t)\| \leq \bar{d}(t)\ \forall t$. Then, there exists a positive constant c_1 such that the impulse response $w(t)$ satisfies $\|w(t)\| \leq c_1\gamma_0 e^{-\gamma_0 t}$ and the following inequalities hold

$$\|w(t) * d(t)\| \leq c_1\gamma_0 e^{-\gamma_0 t} * \bar{d}(t) = c_1\frac{\gamma_0}{s+\gamma_0} * \bar{d}(t), \tag{31}$$

$$\|z(t) - z^0(t)\| \leq c_1\|d_f(t) - d_f^0(t)\|, \quad d_f = (\frac{\gamma_0}{s+\gamma_0})\bar{d}, \tag{32}$$

$$\|z(t)\| \leq c_1 d_f(t) + \exp, \tag{33}$$

where z^0, d_f^0 and "exp" depend on the initial conditions and decay exponentially to zero with rate γ_0.
Proof: The proof follows from a direct extension of the scalar case in [11].

Corollary 2 (Lemma 4) Consider

$$z = G_F(\tau s)G_L(s)d = G_F(\tau s)\frac{1}{s+\alpha}\bar{G}_L(s)d, \tag{34}$$

where G_F, G_L are rational, stable, strictly proper, $\bar{G}_L$ has *positive impulse response*, $\alpha > 0$ is the stability margin of $G_L(s)$. If $\tau \in [0, \bar{\tau}]$ and $\bar{\tau}$ is sufficiently small, there exists $k > 0$ such that (32) and (33) hold with

$$d_f(t) = k\frac{1}{s+\alpha}\bar{d}(t). \tag{35}$$

Proof: The proof follows from a direct extension of the scalar case in [8].

II. Proof of Lemmas and Theorems

A. Proof of Proposition 1

Since $-K$ is Hurwitz, there exists $P = P^T > 0$ and $Q = Q^T > 0$ such that $K^T P + PK = Q$. Thus, consider the quadratic form $V(\varepsilon) = \varepsilon^T P \varepsilon$ which has time derivative bounded by

$$\begin{aligned}\dot V \leq -\rho\lambda_{min}(Q)\|\varepsilon\| + \lambda_{max}(A^T P + PA)\|\varepsilon\|^2 + \\ + 2\sigma_{max}(PK)\|\varepsilon\|\left(\|d\| + \|\pi\|\right) .\end{aligned} \tag{36}$$

Now, choosing ρ as in (10) with

$$\begin{aligned} c_\varepsilon &\geq \max\left\{ \frac{\lambda_{max}(A^T P + PA)}{\lambda_{min}(Q)} + \bar\delta \ , \ 0 \right\} , \\ c_d &\geq 2\ \frac{\sigma_{max}(PK)}{\lambda_{min}(Q)} - 1 , \end{aligned} \tag{37}$$

where the constant $\bar\delta > 0$ provides some desired stability margin, we obtain

$$\dot V \leq -\lambda_{min}(Q)\left[\delta + \bar\delta\|\varepsilon\| - (1 + c_d)Re^{-\lambda t}\right]\|\varepsilon\| . \tag{38}$$

Now, from the Rayleigh-Ritz inequality

$$\lambda_{min}(P)\,\|\varepsilon\|^2 \leq V(\varepsilon) \leq \lambda_{max}(P)\,\|\varepsilon\|^2 , \tag{39}$$

and denoting $c_{Q1} = \lambda_{min}(Q)/\sqrt{\lambda_{max}(P)}$ (> 0), $c_{Q2} = \lambda_{min}(Q)/\lambda_{max}(P)$ (> 0) and $c_D = (1 + c_d)\lambda_{min}(Q)/\sqrt{\lambda_{min}(P)}$ (> 0), inequality (38) can be rewritten as

$$\dot V \leq -\delta\, c_{Q1}\sqrt{V} - \bar\delta\, c_{Q2} V + c_D R e^{-\lambda t}\sqrt{V} . \tag{40}$$

Then, defining $r := \sqrt{V}$, one obtains

$$2\dot r \leq -\delta\, c_{Q1} - \bar\delta\, c_{Q2}\, r + c_D\, R e^{-\lambda t} . \tag{41}$$

Thus from Lemma 3 (see Appendix A), we can conclude that $r(t) \leq [r(0) + cR]e^{-\lambda_1 t}$, where $c > 0$ is an appropriate constant and $\lambda_1 = \min\left(\lambda, \bar\delta\, c_{Q2}/2\right)$. Applying the Rayleigh-Ritz inequality, we finally obtain inequality (11). If $\delta > 0$ in (10), from Lemma 3, one can further conclude that there exists $t_1 < +\infty$ such that $r(t) \equiv 0$, $\forall t > t_1$, hence, the sliding mode at $\varepsilon = 0$ starts in some finite time t_s, $0 \leq t_s \leq t_1$.

B. Proof of Lemma 1

Consider a stabilizable and detectable realization of (13)

$$\dot x = Ax + B(u + d + \pi) , \qquad \varepsilon = Cx . \tag{42}$$

The high frequency gain is $K = CB$. System (42) can be transformed to the regular form [4]

$$\dot x_1 = A_{11}x_1 + A_{12}\varepsilon , \tag{43}$$

$$\dot\varepsilon = A_{21}x_1 + A_{22}\varepsilon + K(u + d + \pi) . \tag{44}$$

The state vector of this realization is $x_e^T = [x_1^T \quad \varepsilon^T]$. The zero dynamics is given by $\dot x_1 = A_{11}x_1$. Since $M(s)$ is minimum phase, A_{11} is Hurwitz. Let γ_0 be the stability margin of A_{11}. Then, according to Lemma 4 applied to (43) written as $x_1 = (sI - A_{11})^{-1}A_{12}\varepsilon$, one has the bound

$$\|x_1(t)\| \leq c_1[\,|\varepsilon_f(t)| + |\varepsilon_f^0(t)|\,] + \|x_1^0(t)\| , \tag{45}$$

with

$$\varepsilon_f = \frac{\gamma_0}{s + \gamma_0} * \|\varepsilon\| ,$$

where ε_f^0 and x_1^0 are exponentially decaying terms due to initial conditions. On the other hand, since $-K$ is Hurwitz, and thus nonsingular, equation (44) can be rewritten as

$$\begin{aligned}\dot\varepsilon = A_{22}\varepsilon + K. \\ .\left[u_1 + d + K^{-1}A_{21}(sI - A_{11})^{-1}A_{12} * \varepsilon + \pi_1\right] ,\end{aligned}$$

where $u_1 = -\rho_1\frac{\varepsilon}{\|\varepsilon\|}$ and $\rho_1 = (1 + c_d)\|d(t)\| + c_\varepsilon\|\varepsilon\| + c_f\frac{\gamma_0}{s+\gamma_0} * \|\varepsilon\|$. The term π_1 takes into account the terms π, x_1^0 and ε_f^0 and is bounded by $\|\pi_1\| \leq [c_1\|x_{ea}(0)\| + R]e^{-\lambda_1 t}$, where $x_{ea} = [x_1^T \ \varepsilon^T \ \varepsilon_f]^T$ is the complete state vector of the closed loop system and $\lambda_1 = \min(\gamma_0, \lambda)$. Thus, from Proposition 1, inequality (17) is satisfied for $\|\varepsilon\|$, provided that $c_d,\ c_\varepsilon,\ c_f \geq 0$ are appropriately chosen. Since, $\|x_1\|$ and $\|\varepsilon_f\|$ satisfy similar inequality, then $\|x_{ea}(t)\|$ also satisfies (17).

Also from Proposition 1, one can further conclude that ε becomes identically zero after some finite time t_s, provided that $\delta > 0$ in (15).

C. Proof of Corollary 1

For simplicity, we will consider a controllable realization. In this case, if there are unobservable states, the element A_{21} of the regular form (43)–(44) is identically zero, i.e., $A_{21} = 0$. Then, the result follows directly from the proof of Lemma 1. In the case of a nonminimal realization which is noncontrolable and/or nonobservable, the proof follows in a similar way, using the Kalman Decomposition.

Now, if $\alpha_i = \alpha$, $(\forall i)$, then $A_{22} = -\alpha I$ satisfies the Lyapunov equation $PA_{22} + A_{22}^T P = -Q_2$, $Q_2 > 0$ for any $P = P^T > 0$. Since the constant c_ε, in Lemma 1, comes from inequality (37) in Proposition 1, one can chose $c_\varepsilon = 0$.

D. Proof of Lemma 2

The time derivative of the quadratic form $V(t) = \hat\varepsilon^T P\hat\varepsilon$ $(P = P^T > 0)$ is given by $\dot V = -2\alpha\hat\varepsilon^T P\hat\varepsilon + [\hat\varepsilon^T PKu + u^T K^T P\hat\varepsilon] + 2\hat\varepsilon^T PKd$. Then, with $u = -\rho\frac{\varepsilon}{\|\varepsilon\|}$, we have that

$$\dot V = -2\alpha V - \rho\frac{\hat\varepsilon^T PK\varepsilon + \varepsilon^T K^T P\hat\varepsilon}{\|\varepsilon\|} + 2\hat\varepsilon^T PKd .$$

Now, define the auxiliary functions

$$\dot V_s = -2\alpha V_s , \qquad \bar V = V_s + \gamma ,$$

where γ is an absolutely continuous function satisfying $\dot\gamma \geq -2\alpha\gamma$ and $\gamma \geq c^2\|\beta + \pi\|^2$ for some positive constant c, yet to be defined. Since $\dot{\bar{V}} = -2\alpha V_s + \dot{\gamma} = -2\alpha(\bar{V} - \gamma) + \dot{\gamma}$, we can write the following *comparison* equation

$$\frac{d}{dt}[\bar{V} - V] = -2\alpha[\bar{V} - V] + \dot{\gamma} + 2\alpha\gamma + + 2\rho\frac{\hat{\varepsilon}^T PK\varepsilon}{\|\varepsilon\|} - 2\hat{\varepsilon}^T PKd\,. \quad (46)$$

Considering γ given by $\gamma(t) = c^2[R\ e^{-min(\alpha,\lambda)t} + \|\beta_t\|_\infty]^2$, it is easy to verify that $V_s(0) > 0$ implies that, $\forall t$,

$$V_s(t) = e^{-2\alpha t}V_s(0) > 0\,, \quad (47)$$

$$\bar{V}(t) > c^2[R\ e^{-min(\alpha,\lambda)t} + \|\beta_t\|_\infty]^2\,. \quad (48)$$

Then, if $\bar{V}(t_0) = V(t_0)$ for any $t_0 \geq 0$, the inequality (48) implies that $V(t_0) > c^2[R\ e^{-min(\alpha,\lambda)t_0} + \|\beta_{t_0}\|_\infty]^2$ ($\forall t_0 \geq 0$). From the Rayleigh-Ritz inequality, $\lambda_{min}(P)\|\hat{\varepsilon}\|^2 \leq V \leq \lambda_{max}(P)\|\hat{\varepsilon}\|^2$, one concludes that

$$\|\hat{\varepsilon}(t_0)\| \geq \frac{c}{\sqrt{\lambda_{max}(P)}}[R\ e^{-\min(\alpha,\lambda)t_0} + \|\beta_{t_0}\|_\infty] \quad (49)$$

and also, for $t = t_0$,

$$\frac{d}{dt}[\bar{V} - V] \geq 2\rho\frac{\hat{\varepsilon}^T PK\varepsilon}{\|\varepsilon\|} - 2\hat{\varepsilon}^T PKd\,. \quad (50)$$

Since $K^T P + PK = Q > 0$, we can rewrite (50) as

$$\frac{d}{dt}[\bar{V} - V] \geq \rho\frac{\varepsilon^T Q\varepsilon - 2\varepsilon^T K^T P(\beta + \pi)}{\|\varepsilon\|} - 2\hat{\varepsilon}^T PKd$$

and consequently, denoting $c_d = 2\sigma_{max}(PK)/\lambda_{min}(Q) - 1\,(\geq 0)$,

$$\begin{aligned}\frac{d}{dt}[\bar{V} - V] &\geq \\ &\geq \lambda_{min}(Q)\left[\rho\frac{\|\varepsilon\| - (c_d + 1)\|\beta + \pi\|}{\|\hat{\varepsilon}\|} - (c_d + 1)\|d\|\right]\|\hat{\varepsilon}\| \\ &\geq \lambda_{min}(Q)\left[\rho\left(1 - (c_d + 2)\frac{\|\beta + \pi\|}{\|\hat{\varepsilon}\|}\right) - (c_d + 1)\|d\|\right]\|\hat{\varepsilon}\|\,.\end{aligned}$$

Then, choosing $c \geq (1 + k)(c_d + 2)\sqrt{\lambda_{max}(P)}$ in (49) for some $k > 0$, one has

$$\frac{\|\beta(t_0) + \pi(t_0)\|}{\|\hat{\varepsilon}(t_0)\|} \leq \frac{1}{(1 + k)(c_d + 2)}\,. \quad (51)$$

Therefore, we have for $t = t_0$,

$$\frac{d}{dt}[\bar{V} - V] \geq \lambda_{min}(Q)\left[\rho\frac{k}{1 + k} - (c_d + 1)\|d\|\right]\|\hat{\varepsilon}\|\,.$$

Now, if we choose $\rho(t)$ such that $\rho(t) \geq (\hat{c}_d+1)\|d(t)\|\,, \quad \forall t\,,$ where $\hat{c}_d = (1 + k^{-1})(c_d + 1) - 1 > 0$, we have that, for $t = t_0$, $\frac{d}{dt}[\bar{V} - V] \geq 0$. Thus using the *Comparison Theorem* in [2, Theorem 7], we conclude that $\dot{V}(t_0) \leq \dot{\bar{V}}(t_0)$ implies $V(t) \leq \bar{V}(t), \forall t \geq t_0$. Then, from the Rayleigh-Ritz inequality and since $\bar{V}(t) = e^{-2\alpha t}V_s(0)+\gamma(t)$, we obtain inequality (21), with $c_1 = \sqrt{\lambda_{max}(P)/\lambda_{min}(P)}$ and, without loss of generality, $t_0 = 0$.

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