Paper: Output Tracking of Uncertain Nonlinear Systems with Unknown Control Direction.

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A. Proof of Proposition 1

In what follows, k_i denote positive constants that depends only on the plant-controller parameters and $\Psi_i(\cdot)$ denote functions of class $\mathcal{K}_\infty$. Since $f_d \leq \|(\bar{\beta}_\mathcal{U})_t\|$, from (23), (24) and (26), one has

$$\|(\varepsilon_0)_{t,t_1}\| \leq |\varepsilon_0(t_p)| + a(k) + 3\|(\bar{\beta}_\mathcal{U})_t\|\,, \;\; \forall t \in [t_1, t_M)\,, \quad (39)$$

where $k \geq 1$, $p = \text{argmax}_{i\in\{1,2,\ldots,k\}}|\varepsilon_0(t_i)|$ and $t \in [t_k, t_{k+1}]$.

From (15), one can conclude that $\|(e_F^0)_{t,t_1}\| \leq k_1|z(0)|$, with z defined in (27). From (13) and reminding that $\bar{e}_0 := \rho^* ML(s)\,[u - u^*]$, one has $\bar{e}_0 = \varepsilon_0 - \beta_\mathcal{U} - e_F^0$. Thus, the following inequality holds $\forall t \in [t_1, t_M)$

$$\|(\bar{e}_0)_{t,t_1}\| \leq |\bar{e}_0(t_p)| + a(k) + 2k_1|z(0)| + 5\|(\bar{\beta}_\mathcal{U})_t\|\,. \quad (40)$$

Now, since X_e is the state of a stable non-minimal realization of the transfer function $ML(s) = k_m/(s + a_m)$; $k_m, a_m > 0$, it is possible to linearly transform X_e to a new state $\bar{X}_e = PX_e = \begin{bmatrix} \bar{e}_0 & \bar{X}_{e2}^T \end{bmatrix}^T$, where $\bar{X}_{e2}$ is an exponentially decaying term [10]. Moreover, since x_f is driven by the tracking error $e_0 = h_c^T X_e$, and taking into account (28), an upper bound similar to (40) is also valid for z (27), i.e., the state z satisfies $\forall t \in [0, t_M)$

$$\|(z)_t\| \leq k_2|z(0)| + k_3|z(t_p)| + k_4 a(k) + k_5\|(\bar{\beta}_\mathcal{U})_t\| + \mathcal{O}(\tau)\,. \quad (41)$$

Now, from the small norm property of $W_\beta(s,\tau)$ (18), since $f(t)$ is given by (12) with ω affinely bounded by $|X_e|$ [9] and $\|X_e\| \leq |z|$, then one has

$$\|(\bar{\beta}_\mathcal{U})_t\| \leq \tau\Psi_1(\|(z)_t\|) + \tau k_6\,. \quad (42)$$

Now, given $R > 0$ and $0 < R_0 < R$, then for some $t^* \in [0, t_M)$, which is *independent of* τ, and $|z(0)| < R_0$ one has $|z(t)| < R$ for $t \in [0, t^*)$. Then,

$$\Psi_1(|z|) \leq k_1^R|z|\,, \;\; \forall |z| < R\,,$$

with the positive constant k_1^R possibly dependent on R. Moreover, from (41) and (42), for $\tau < 1/(k_1^R k_5)$, we get $\forall t \in [0, t_M)$

$$\|(z)_t\| \leq k_7|z(t_p)| + k_8 a(k) + k_9|z(0)| + \mathcal{O}(\tau)\,. \quad (43)$$

Then, noting that $|z(t_p)| \leq \sum_{i=1}^k |z(t_i)|$, the following recursive inequality follows

$$|z(t_{k+1})| \leq k_7 \sum_{i=1}^{k} |z(t_i)| + k_8 a(k) + k_9|z(0)| + \mathcal{O}(\tau)\,, \quad (44)$$

whereby (29) results. ∎

B. Proof of Theorem 1

The monitoring function (24) has to stop switching after a finite number $k = k^*$ of switchings. The proof is obtained by contradiction. Suppose that u (11) switches between u^+ and u^- without stopping. Then, $a(k)$ in (23) increases unboundedly as $k \to \infty$. Thus, there is a finite value k_1 such that

$$a(k_1) > 2R_a e^{\bar{\lambda}_a \bar{t}_e} \quad (45)$$

and the control direction is correctly estimated. In this case, $|\xi(t)| < \varphi_m(t), \forall t \geq t_{k_1}$, where

$$\begin{aligned} \varphi_m(t) \;\; &= \;\; (|\varepsilon_0(t_{k_1})| + |\bar{\beta}_\mathcal{U}(t_{k_1})|)e^{-a_m(t-t_{k_1})} + \\ &+ \;\; a(k_1)e^{-\lambda_c t} + 2f_d(t)\,, \end{aligned} \quad (46)$$

is the monitoring function (24) valid $\forall t \geq t_{k_1}$ and

$$\begin{aligned} \xi(t) \;\; &:= \;\; (|\varepsilon_0(t_{k_1})| + |\beta_\mathcal{U}(t_{k_1})|)e^{-a_m(t-t_{k_1})} + \\ &+ \;\; (2R_a e^{\bar{\lambda}_a \bar{t}_e})e^{-\bar{\lambda}_a t} + 2f_d(t) \end{aligned} \quad (47)$$

is a valid upper bound for $|\varepsilon_0(t)|$ if $\text{sgn}(k_p)$ is correct. Hence, no switching will occur after that, which leads to a contradiction. Therefore, the monitoring function has to stop switching after some finite $k = k^*$. Now, from (45), it is not difficult to conclude that k^* can be related to $|z(0)|$, reminding that $R_a \leq k_a|z(0)|$ by definition. In fact, one can write

$$k^* \leq \mathcal{V}_k(R_0) + k_0\,, \quad (48)$$

where $k_0 > 0$ is a constant and $\mathcal{V}_k \in \mathcal{K}_\infty$.

Now, from Proposition 1, for τ sufficiently small, one can conclude that the full state error z is uniformly bounded by

$$|z(t)| \leq \mathcal{V}_z(R_0) + c_z\,, \;\; \forall t \in [0, t_M)\,, \quad (49)$$

where, $c_z > 0$ is a constant and $\mathcal{V}_z \in \mathcal{K}_\infty$. Given $R > c_z$, there exists $R_0 > 0$ such that for $|z(0)| < R_0$ then one has $|z(t)| < R$, $\forall t \in [0, t_M)$. Thus, stability with respect to the ball of radius c_z is guaranteed for initial conditions in the R_0-ball. This implies that $z(t)$ is uniformly bounded and cannot escape in finite time, i.e., $t_M = +\infty$. Since R_0 can be chosen arbitrarily large provided τ is chosen sufficiently small, semi-global stability is concluded.

If the control direction is correctly found at $k = k^*$, then $|z(t)| \to \mathcal{V}(\tau)$ exponentially ($\mathcal{V} \in \mathcal{K}$), as $t \to +\infty$, according to [11, Theorem 3]. Otherwise, the control pursues with wrong control direction estimate $\forall t > t_{k^*}$ while all signals remain uniformly bounded due to (49). Since we have chosen a modulation function capable of making the closed loop unstable if wrong control direction estimate is applied, one can show that there exists a sign indefinite quadratic function $V(z)$ which has positive time derivative outside a compact set around the error space origin. According to a stability analysis similar to that of Cetaev's Instability Theorem [8], this implies that the system must enter a residual set where $|X_e(t)| < \mathcal{V}(\tau)$ after some finite time. A rigorous proof follows closely the method of [16]. In addition, reminding that the state x_f is driven by the signal $\bar{e}_0 = h_L^T X_e$, then the convergence of X_e implies $|x_f(t)|, |z(t)| < \mathcal{V}(\tau)$, $\forall t$ after some finite time. ∎

C. Proof of Corollary 1

The hybrid lead filter only introduces a disturbance β_α [5] which is norm-bounded by a design constant of order $\mathcal{O}(\tau)$, *modulo* decaying exponential terms which can be embedded in e_F^0 (15). This constant bound can be simply added to the bound of $\bar{\beta}_\mathcal{U}$ given in (18). The monitoring function is redefined in an appropriate way in order to monitor the perturbed auxiliary signal $\tilde{\varepsilon}_0$. The exact differentiator will eventually take over providing the exact estimate of the ideal sliding variable $\bar{e}_0$, i.e. $\tilde{\varepsilon}_0 = \bar{e}_0$, since the error state enters the residual set (*Theorem 1*). After that the system becomes exactly a relative degree one case, with sliding variable $\bar{e}_0$.

To conclude the demonstration we now prove that the control direction is correctly estimated after the last switching at $k = k^*$. This can be shown by contradiction. Suppose we ended up with an incorrect control direction estimate. Then, the equation for the ideal sliding variable $\bar{e}_0$ can be written as:

$$\dot{\bar{e}}_0 = a_m\bar{e}_0 + |k_p|(f(t)\text{sgn}(\bar{e}_0) - u^\dagger) + \pi,$$

where a_m is a positive constant. In this case, due to the modulation function (12), $\bar{e}_0$ diverges as $t \to \infty$ for all initial conditions except, possibly, for a set of zero measure. Hence, $\bar{e}_0$ would not remain in the residual set of *Theorem 1*, leading to a contradiction. The same conclusion can be achieved by using a Cetaev's Theorem argument. ∎

D. Proof of Lemma 1

Consider a detectable and stabilizable realization of (31)

$$\dot{x} = Ax + B(u - u^*)\,, \qquad e_0 = Cx\,. \tag{50}$$

The high frequency gain is $K_p = CB$. System (50) can be transformed to the *regular form*

$$\dot{x}_1 = A_{11}x_1 + A_{12}e_0\,, \tag{51}$$

$$\dot{e}_0 = A_{21}x_1 + A_{22}e_0 + K_p(u - u^*)\,. \tag{52}$$

The state vector of this realization is $x_e^T = [x_1^T \quad e_0^T]$ and A_{11} is Hurwitz. For simplicity, we will consider a controllable realization. In this case, if there are unobservable states, the element A_{21} of the regular form (51)–(52) is identically zero, i.e., $A_{21} = 0$. In the case of a nonminimal realization which is noncontrollable and/or nonobservable, the proof follows in a similar way, using the Kalman Decomposition.

First, one proves that the switching stops after a finite number of switchings, since for some finite k^* the term $(k^*+1)e^{-t/(k^*+1)}$ of (33) will be a bound for $|\pi(t)|$ (32) such that $|e_0(t)| < \varphi_M(t), \forall t \geq t_{k^*}$, and will then switch at most one more cycle throughout the index set $\mathcal{Q}$. Then, one concludes (independently of whether a Hurwitz matrix $-K_pS_q$ is selected at $k = k^*$ or not) that $e_0(t)$ will converge to zero, at least exponentially, since φ_M converges to zero exponentially.

In addition, reminding that the state x_1 is driven by e_0, then the convergence of e_0 implies $|x_1(t)|, |x_e(t)| \to 0$, $\forall t \geq t_{k^*}$. Also from [13, Proposition 1], one can further conclude that e_0 becomes identically zero after a finite time t_s, provided that $\delta > 0$ in (35).

As we can see in the SISO case, it is not difficult to conclude that k^* can be related to $R_0 := |x_e(0)|$. In fact, one can write

$$k^* \leq \mathcal{V}_k(R_0) + k_0\,, \tag{53}$$

where $k_0 > 0$ is a constant and $\mathcal{V}_k \in \mathcal{K}_\infty$. Moreover, we can conclude that

$$|x_e(t)| \leq \mathcal{V}(R_0) + c\,, \tag{54}$$

where, $c > 0$ is a constant and $\mathcal{V} \in \mathcal{K}_\infty$. Given $R > c$, there exists $R_0 > 0$ such that for $|x_e(0)| < R_0$ then one has $|x_e(t)| < R$, $\forall t \geq 0$. Thus, stability with respect to the ball of radius c is guaranteed for initial conditions in the R_0-ball. Since R_0 can be chosen arbitrarily large, global stability is concluded. ■

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