APPENDIX

Title: Facing the Global Tracking Output-feedback Sliding Mode Control Problem via Norm Estimators and Variable High Gain Observer.

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A. Geometric Conditions for Normal Form

In order to consider explicitly the time dependence of $f(x,t)$ in (1)–(1), let: $\beta_k := L_f\beta_{k-1} + \frac{\partial \beta_{k-1}}{\partial t} + \frac{\partial [L_f^{k-1}h]}{\partial t}$, for $k \in \{1,\dots,\rho\}$, where $\beta_0 := 0$. A sufficient condition to assure that the *time-varying* plant (1)–(2) is transformable to the normal form is given by: $L_g[L_f^k h + \beta_k] \equiv 0$ $(k \in \{0,\dots,\rho-2\})$, where Lie derivative of a function h along a vector field f is denoted by $L_f h$, as in [23, pp. 510]. In this case, the transformation $T(x,t) = [\eta^T \; T_\xi^T(x,t)]$ is such that $T_\xi := \left[\begin{array}{cccc} L_f^0 h + \beta_0 & L_f h + \beta_1 & \dots & L_f^{\rho-1}h + \beta_{\rho-1}\end{array}\right]^T$. In addition, the plant HFG $k_p(x,t) = L_g[L_f^{\rho-1}h + \beta_{\rho-1}]$, the input disturbance $d(x,t) = (L_f^\rho h + \beta_\rho)/k_p$ and T must satisfy Assumption 1.

B. Norm Observer

In this section, we consider systems in the form (44)–(45) satisfying (C0) and (C1) in Section VIII. In what follows, we give the steps to obtain the norm observer (47)–(49), according to Definition 1.

1) Norm bound for η: obtaining c_0 and φ_1 in (48): From (C1), the function α_1 is stiffening. It guarantees that $\alpha_1(\sigma) > \lambda\sigma$, $\forall \sigma > \epsilon$, for any $\epsilon > 0$ and $0 < \lambda < \alpha_1(\epsilon)/\epsilon$. Moreover, from (46), one can write $\dot V \leq -\alpha_1(V) + \varphi_\eta(y,t)$ or, equivalently, $\dot V \leq -\lambda V + [\lambda V - \alpha_1(V)] + \varphi_\eta(y,t)$. Now, given any V, either $V \leq \epsilon$ or $V > \epsilon$. Hence, either $\dot V \leq -\lambda V + [\lambda\epsilon + \alpha_1(\epsilon)] + \varphi_\eta$ or $\dot V \leq -\lambda V + \varphi_\eta$, leading to the conclusion that $\dot V \leq -\lambda V + [\lambda\epsilon + \alpha_1(\epsilon)] + \varphi_\eta$. Therefore, by using comparison theorems [23], one has

$$V \leq e^{-c_0 t} * \varphi_1(y,t) + V(\eta(0))e^{-c_0 t}\,,$$

where $\varphi_1 = \varphi_\eta + c_0\epsilon + \alpha_1(\epsilon)$, $c_0 = \lambda$ are *known* and the operator $*$ denotes pure convolution. Reminding that $\underline{\lambda}|\eta|^2 \leq V$, then one can obtain $|\eta| \leq \sqrt{|\omega_{21}|/\underline{\lambda}} + \pi_0$, with ω_{21} in (48) and π_0 is an exponentially decaying term depending on $|\eta(0)|$ and $|\omega_{21}(0)|$.

2) Norm bound for v: obtaining φ_2 and φ_3 in (49): It is useful to rewrite (45) in the compact form

$$\dot v = A_\rho v + B_\rho k_u u + \phi(x,t)\,, \tag{53}$$

where (A_ρ, B_ρ) is the Brunovsky's canonical pair and apply the change of variable $\bar v = v - B_\rho k_u \tau_1 \omega_1$ to obtain:

$$\dot{\bar v} = A_\rho \bar v + B_\rho k_u \omega_1 + \phi\,.$$

By observability of the pair (A_ρ, C_ρ), where $C_\rho = [1 \;\; 0 \;\; \dots \;\; 0]$, there exist a matrix $P = P^T > 0$ and an arbitrary column vector L satisfying $A_L^T P + P A_L = -I$, where $A_L = A_\rho - LC_\rho$ is a Hurwitz matrix.

Now, with $T := \text{diag}(1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{\rho-1})$ and any given constant $\varepsilon > 0$, the following properties can be checked: (i) $TA_\rho T^{-1} = \varepsilon^{-1}A_\rho$, (ii) $C_\rho T^{-1} = C_\rho$ and (iii) $TB_\rho = B_\rho \varepsilon^{\rho-1}$. Then, adding and subtracting the term $(\varepsilon T)^{-1}LC_\rho \bar v$ to the $\bar v$-dynamics, one can write $\dot{\bar v} = [A_\rho - (\varepsilon T)^{-1}LC_\rho]\bar v + B_\rho k_u \omega_1 + \phi + (\varepsilon T)^{-1}Ly$. Moreover, applying the transformation $\vartheta = T\bar v$ and the above properties (i)–(iii), one can also write

$$\dot\vartheta = \varepsilon^{-1}A_L\vartheta + B_\rho\varepsilon^{\rho-1}k_u\omega_1 + \varepsilon^{-1}Ly + T\phi\,.$$

The key step is to note that, due to the triangularity condition (C0):

$$|T\phi| \leq k_\vartheta \varphi_r |\vartheta| + \varphi_\vartheta\,,$$

where k_ϑ is ε-independent. Then, by using the Dini derivative[4] and the bounding function Ψ_v given in (C0), the time derivative of $V := (\vartheta^T P \vartheta)^{1/2}$ along the solution of the ϑ-dynamics satisfies

$$\dot V \leq -\frac{c_1}{\varepsilon}V + c_2\varphi_r V + \bar\varphi_1(\omega_{21}, \omega_1, y, t, \varepsilon) + \pi_1\,,$$

where π_1 is a exponentially decaying term and the non-negative function $\bar\varphi_1$ and the non-negative constants c_1, c_2 are all *known* and satisfy $c_1 \leq 1/(2\lambda_M[P])$, $c_2 \geq |P|k_\vartheta/\lambda_m[P]$ and $[|B_\rho\varepsilon^{\rho-1}k_u\omega_1 + \varepsilon^{-1}Ly| + \varphi_\vartheta]c_3 \leq \bar\varphi_1 + \pi_1$, with $c_3 \geq |P|/\sqrt{\lambda_m[P]}$.

Therefore, given any V, either

$$V \leq \bar\varphi_1 \quad \text{or} \quad \dot V \leq -\frac{c_1}{\varepsilon}V + c_2\varphi_r V + V + \pi_1\,. \tag{54}$$

Now, let

$$\bar\varphi_4(\omega_{21}, y, t) := \varphi_2(\omega_{21}) + \varphi_3(y,t)\,, \tag{55}$$

with the non-negative functions φ_2, φ_3 in (49) to be determined. Then, (49) can be rewritten as

$$\dot\omega_{22} = -\frac{1}{\tau_2}\gamma(\omega_{22}) + \bar\varphi_4\,, \tag{56}$$

with $\gamma(\sigma) := 1 - e^{-\sigma}$. Hence, by using the bounding function Ψ_r, given in (C0), we must choose $\bar\varphi_4$ in (55) (and the functions φ_2, φ_3) in order to satisfy:

$$c_2\varphi_r + 1 \leq \bar\varphi_4(\omega_{21}, y, t)\,.$$

Norm Bound for v

The norm bound for the v-subsystem can be obtained considering two cases for the growth rate $\varphi_r(|\eta|, y, t)$: $\varphi_r > k_r$ and $\varphi_r \leq k_r$, where $k_r = 3/(c_2\tau_2)$ and τ_2 is the positive design constant in (56).

Case 1: In this case, one has $3/\tau_2 \leq c_2 k_r + 1 \leq c_2\varphi_r + 1 \leq \bar\varphi_4$. Thus, one can verify that

$$\gamma(\sigma) \leq 2 \leq \tau_2\bar\varphi_4 - 1\,, \quad \forall\sigma\,. \tag{57}$$

Now, let $W := \ln(V+1)$ [19]. Then, $\dot W = \dot V/(V+1)$ and, from (54), one can write

$$V \leq \bar\varphi_3 \quad \text{or} \quad \dot W \leq -\frac{1}{\tau_2}\gamma(W) + \bar\varphi_4 + \pi_1\,, \tag{58}$$

[4]To avoid the Dini derivative we could have used the relationship $ab \leq a^2 + b^2$, valid $\forall a, b > 0$, at the expense of some conservatism.

with $\varepsilon = c_1\tau_2$ and $\bar{\varphi}_3 := \bar{\varphi}_1|_{\varepsilon=c_1\tau_2}$. Note that we have used the relationship $V/(V+1)\,,\ 1/(V+1) \leq 1$.

Now, given any W, we have two possibilities: $W < \omega_{22}$ or $W \geq \omega_{22}$. Considering the later case, one can write $-\gamma(\omega_{22}) \geq -\gamma(W)$, since γ is a increasing function. Therefore, from (58) and (56), one has $\dot{\omega}_{22} \geq \dot{W} - \pi_1$. In addition, from (57), $\dot{\omega}_{22}$ also satisfies $\dot{\omega}_{22} \geq 1/\tau_2$. Consequently, adding the last two inequalities one has

$$\dot{W} - 2\dot{\omega}_{22} \leq -\frac{1}{\tau_2} + \pi_1 \,.$$

Now, recall that $\pi_1 = \beta_1 e^{-\lambda_1 t}$ and let $\bar{W} = W + \pi_1/\lambda_1$, for some positive constant λ_1 and some $\beta_1 \in \mathcal{K}_\infty$. Then, one has $\dot{\bar{W}} - 2\dot{\omega}_{22} \leq -\frac{1}{\tau_2}$, from which one can conclude that, $\bar{W} \leq 2\omega_{22} - t/\tau_2 + |\bar{W}(0) - 2\omega_{22}(0)|$. Note that, it is always possible to find an exponential decaying term which is an upper bound for the above affine time function, i.e., $-t/\tau_2 + |\bar{W}(0) - 2\omega_{22}(0)| \leq \pi_2$, where $\pi_2 := \beta_2(|\bar{W}(0)| + |\omega_{22}(0)|)e^{-\lambda_2 t}$, with $\beta_2 \in \mathcal{K}_\infty$ and some constant $\lambda_2 > 0$. Finally, given W, one can conclude that $W \leq 2|\omega_{22}| + \pi_2 + \pi_1/\lambda_1$ and, by using comparison theorems [23] and recalling that $V = e^W - 1$ one can write

$$V \leq e^{2|\omega_{22}|} + \pi_3 \,, \tag{59}$$

where π_3 is an exponential decaying term.

Case 2: Assume now that $\varphi_r \leq k_r$ and set $\varepsilon = c_1/(c_2 k_r + 2)$ in (54). Then, one can write:

$$V \leq \bar{\varphi}_2 \quad \text{or} \quad \dot{V} \leq -V + \pi_1 \,, \tag{60}$$

where $\bar{\varphi}_2 = \bar{\varphi}_1|_{\varepsilon=c_1/(c_2 k_r+2)}$. In this case, adding the two upper bounds obtained from (60) one can write

$$V \leq \bar{\varphi}_2 + \pi_4 \,, \tag{61}$$

where π_4 is an exponential decaying term. Then, from (59) and (61) one has

$$V \leq e^{2|\omega_{22}|} + \bar{\varphi}_1(\omega_{21}, \omega_1, y, t, \varepsilon) + \pi_5 \,, \tag{62}$$

with $\varepsilon = c_1/(3/\tau_2 + 2)$ and using the Rayleigh's inequality one can obtain an upper bound for v.

Finally, putting together the norm bounds for v and η we obtain the non-negative function φ_4 and the non-negative constants in (50).

C. Proof of Lemma 1

First, applying the coordinate transformation $\xi_{en} = T_n \xi_e$, where $T_n := [\ I \quad S^T\ ]^T$, system (12) can be rewriting into the normal form and one can conclude that (12) is OSS w.r.t. the output $S\xi_e$, i.e., ξ_e satisfies

$$|\xi_e| \leq k_1 |S\xi_e| + \pi_1 \,,$$

where k_1 is a positive constant and $\pi_1 = \beta_1(|\xi_e(0)|)e^{-\lambda_m t}$, with some $\beta_1 \in \mathcal{K}_\infty$ and $0 < \lambda_m < \lambda_m[A_m]$. Given any $\tilde{\xi}_e$, either $|S\xi_e| \leq |S\tilde{\xi}_e|$ or $|S\xi_e| > |S\tilde{\xi}_e|$. Hence, either $|S\xi_e| \leq |\ S\tilde{\xi}_e|$ or $\text{sgn}(\hat{\sigma}) = \text{sgn}(S\xi_e)$. Consider the later case. Then, by using the storage function $V = \xi_e^T P \xi_e$, where $P = P^T > 0$ is the solution of $A_m^T P + P A_m = -I$, one can conclude that the time derivative of V along the solutions of (12) satisfies

$$\dot{V} \leq -|\xi_e|^2 - 2k_p |S\xi_e| [\varrho - |d_e|] \,.$$

Thus, since ϱ in (18) satisfies (17), i.e., $\varrho > |d_e|$, then one has $\dot{V} \leq -|\xi_e|^2$, which leads to the conclusion that $|S\xi_e| \leq |S\tilde{\xi}_e| + \pi_2$ and, consequently, the ξ_e-dynamics is ISS w.r.t. $\tilde{\xi}_e$. ■

D. Proof of Theorem 1

[STEP-1]: From Definition 1, Assumption 1 and (40), one can verify that $|z(t)| \leq \beta_1(|z(0)|) + k_1\,,\ \forall t \in [0, t_\mu]$, where $\beta_1 \in \mathcal{K}_\infty$ and $k_1 \geq 0$ is a constant.

[STEP-2]: Consider the ζ-dynamics (27) and the storage $V = \zeta^T P \zeta$, where $P = P^T > 0$ is the solution of $A_o^T P + P A_o = -I$. Then, the time derivative of V along the solutions of (27) satisfies $\mu\dot{V} = -|\zeta|^2 + (\dot{\mu})[2\zeta^T P \Delta \zeta] + (\mu\nu)[2\zeta^T P B_\rho]$. Now, designing μ to satisfy (P0)–(P2), (41) holds and the following inequality is valid $\forall t \in [t_\mu, t_M)$: $\mu\dot{V} \leq -|\zeta|^2 + \mathcal{O}(\bar{\mu})k_1|\zeta|^2 + \mathcal{O}(\bar{\mu})k_2|\zeta|$, where $k_1 := 2|P||\Delta|$ and $k_2 := 2|P||B_\rho|$. Moreover, since $ab < a^2 + b^2$, for any positive real numbers a, b, then

$$\mu\dot{V} \leq -[1 - \mathcal{O}(\bar{\mu})k_1 - \mathcal{O}(\bar{\mu})]|\zeta|^2 + \mathcal{O}(\bar{\mu}) \,,$$

from which one can conclude that $\mu\dot{V} \leq -\lambda_1 V + \mathcal{O}(\bar{\mu})$, with an appropriate constant $\lambda_1 > 0$. Now, either $V \leq 2\mathcal{O}(\bar{\mu})/\lambda_1$ or $\mu\dot{V} \leq -\lambda_1 V/2$. Consider the later case. Since $\mu < \bar{\mu}$, then one has $\dot{V} \leq -\lambda_1 V/(2\bar{\mu})$. Hence, one can conclude that $|\zeta|\,, |\tilde{\xi}_e| \leq \beta_2(|\zeta(0)|)e^{-\lambda_2 t} + \mathcal{O}(\bar{\mu})$, $\forall t \in [t_\mu, t_M)$, with an appropriate constant $\lambda_2 > 0$ and some $\beta_2 \in \mathcal{K}_\infty$. In the last inequality, the norm bound for $\tilde{\xi}_e$ was obtained by noting that $\tilde{\xi}_e = T_\mu^{-1}\zeta$ implies $|\tilde{\xi}_e| \leq |\zeta|$, since $|T_\mu^{-1}| \leq 1$ for $\mu < 1$.

[STEP-3]: Applying Lemma 1, there exists an ISS Property from $|\tilde{\xi}_e|$ to ξ_e and, considering the norm bound given in STEP-1, one can further concluded that $|\xi_e|, |z(t)| \leq [\beta_3(|z(0)|) + k_3]e^{-\lambda_3 t} + \mathcal{O}(\bar{\mu})$, $\forall t \in [0, t_M)$, with an appropriate constants $\lambda_3 > 0$, $k_3 \geq 0$ and some $\beta_3 \in \mathcal{K}_\infty$. Thus, $|z(t)|$ cannot escape in finite time and it is uniformly bounded in $\mathcal{I} := [0, t_M)$ (UB$\mathcal{I}$).

[STEP-4]: Since $z(t)$ is UB$\mathcal{I}$, then ξ_e, $\sigma = S\xi_e$, ζ and $\xi = \xi_e + \xi_m$ are UB$\mathcal{I}$ and, from Assumption 2, $\eta, \bar{x}$ are also UB$\mathcal{I}$. In addition, according to the lower bound for $|T(x,t)|$ given in Assumption 1 one has that x UB$\mathcal{I}$. Thus, the bounding functions given in Assumption 1 guarantee that d, k_p, d_e are also UB$\mathcal{I}$. Now, rewriting (12) into the normal form one can write $\dot{\sigma} = -\lambda_4\sigma + k_4(u + d_e)$, for some constants $\lambda_4, k_4 > 0$. Moreover, by linearity of the solution of the last equation, one can further write $\sigma = \sigma_1 + \sigma_2$, where $\dot{\sigma}_1 = -\lambda_4\sigma_1 + k_4 u$ and $\dot{\sigma}_2 = -\lambda_4\sigma_2 + k_4 d_e$, with appropriate initial conditions. Thus, since σ and d_e are UB$\mathcal{I}$ so are σ_2 and σ_1. Then, any signal satisfying $\dot{\sigma}_3 = -\lambda_5\sigma_3 + k_5 u$, where $\lambda_5, k_5 > 0$ are constants, is also UB$\mathcal{I}$, in particular, ω_1 defined in (3). Since y, ω_1 is UB$\mathcal{I}$ and φ_o is piecewise continuous in its arguments then the ω_2-dynamics, in Definition 1, cannot escape in finite time. Finally, one can conclude that all system signals cannot escape in finite time, i.e., $t_M \to \infty$. Now, from STEP-3, one can directly verify that the error system is GAS with respect

to the compact set $\{z : |z| \leq b\}$ and ultimate exponential convergence of $z(t)$ to a residual set of order $\mathcal{O}(\bar{\mu})$.

Closed Loop Signals Boundedness: One can further conclude, subsequently, that $|\xi|, y$, $|\eta|, |x|, \sigma_1$ and ω_1 converge to a set of order $\mathcal{O}(|r| + k_5)$ after some finite time, where k_5 is a positive constant depending on the time-varying disturbances. Then, there exists τ_2 sufficiently small and independent of the initial conditions, which assures that ω_2 is bounded after some finite time. Finally, one can conclude that all system signals are UB $\forall t$. ∎

E. Proof of Corollary 1

Recalling that $A_\rho = A_m - B_\rho K_m$, $\hat{\xi} = \hat{\xi}_e + \xi_m$, $\hat{\xi} = \xi_e + \xi_m - \tilde{\xi}_e$, $\hat{\xi}_e = \xi_e - \tilde{\xi}_e$ and $\tilde{\xi}_e = T_\mu^{-1}\zeta$, then from (22) one can write $\dot{\hat{\xi}}_e = A_m \hat{\xi}_e + B_\rho u + \varsigma_m + \varsigma_e$, where $\varsigma_m = -B_\rho(K_m \xi_m + k_m r)$ and $\varsigma_e = (B_\rho K_m + H_\mu L_o C_\rho)(\tilde{\xi}_e - \xi_e) + H_\mu L_o e$. Note that, from Theorem 1, all system signals are uniformly bounded and $z(t) \to \mathcal{O}(\bar{\mu})$. Then, there exists a finite time $T_1 > 0$ such that $|\varsigma_e| \leq \delta_1$, $\forall t \geq T_1$, for any $\delta_1 > 0$. Now, consider the storage function $V = \hat{\xi}_e^T P \hat{\xi}_e$, where $P = P^T > 0$ is the solution of $A_m^T P + P A_m = -Q$, where $Q = Q^T > 0$ and $P B_\rho = S^T$ (recall that (A_m, B_ρ, S) is strictly positive real). Then, computing $\dot{V}$ along the solutions of the $\hat{\xi}_e$-dynamics, one can verify that the condition for the existence of sliding mode $\hat{\sigma}\dot{\hat{\sigma}} < 0$ is verified for some finite time $T_2 \geq T_1$ provided that $\varrho \geq \varsigma_m + \delta$, where $\delta > 0$ is an arbitrary constant. ∎