Paper: Sliding Mode Control of Uncertain Nonlinear Systems with Arbitrary Relative Degree and Unknown Control Direction.

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APPENDIX

A. Proof of Proposition 1

In what follows, k_i denote positive constants that depends only on the plant-controller parameters and $\Psi_i(\cdot)$ denote functions of class $\mathcal{K}_\infty$. Consider the solution of the state equation (4) and $\forall t \in [0, t_M)$, i.e,

$$X_e(t) = k^*(sI - A_c)^{-1} b_c * [u(t) - \bar{u}(t)] + e^{A_c t} X_e(0) . \quad (64)$$

Applying [25, Lemma 2] (considering initial time $\bar{t}_0 = 0$) to (64), follows that ($\forall t \in [0, t_M)$),

$$|X_e(t)| \leq c_1 e^{-\gamma t} * \hat{u}(t) + c_2 e^{-\underline{\lambda}_c t} |X_e(0))| , \quad (65)$$

with $0 < \underline{\lambda}_c < \lambda_c$ where λ_c is the stability margin of A_c, $\gamma > 0$ is lower than the stability margin $\gamma_c > 0$ of the transfer function $(sI - A_c)^{-1} b_c$ and $c_1, c_2 > 0$ are positive appropriate constants, all obtained from [25, Lemma 2] and $|u(t) - \bar{u}(t)| \leq \hat{u}(t)$. Noting that, $\lambda_c \leq \gamma_c$, then one can choose $\gamma = \underline{\lambda}_c$, resulting in the following upper bound, valid $\forall t \in [0, t_M)$,

$$|X_e(t)| \leq c_1 e^{-\underline{\lambda}_c t} * \hat{u}(t) + c_2 e^{-\underline{\lambda}_c t} |X_e(0))| . \quad (66)$$

From Assumptions (A5)–(A6) $\bar{u}$ satisfies (11), thus from Assumption (A7), one has $\forall t \in [0, t_M)$,

$$|u - \bar{u}| \leq \Psi_1(|X_e|) + e^{-\gamma_d t} * \Psi_2(|X_e|) + k_1 := \hat{u} . \quad (67)$$

Consequently, from (66) and (67) and using the *Comparison Theorem* [15], follows that

$$|X_e(t)| \leq \bar{x}_e(t) \leq \bar{\bar{x}}_e(t) , \ \forall t \in [0, t_M) , \quad (68)$$

where $\bar{x}_e(t) := e^{-\lambda_1 t} * [\Psi_3(|X_e(t)|) + k_2] + c_2 e^{-\lambda_1 t} |X_e(0))|$ is the solution of the following differential equation ($\forall t \in [0, t_M)$)

$$\dot{\bar{x}}_e = -\lambda_1 \bar{x}_e + [\Psi_3(|X_e|) + k_2] , \ \bar{x}_e(0) := c_2 |X_e(0))| , \quad (69)$$

and $\bar{\bar{x}}_e$ is the solution of

$$\dot{\bar{\bar{x}}}_e = -\lambda_1 \bar{\bar{x}}_e + [\Psi_3(\bar{\bar{x}}_e) + k_2] , \ \bar{\bar{x}}_e(0) := c_2 |X_e(0))| . \quad (70)$$

Now, from (70) and (68) one can conclude that $\forall R_0 > 0$, there exist $\forall R \geq R_0$ and some $t^* \in (0, t_M)$, which is *independent of* τ (the time constant of the lead filter), such that $|X_e(t)| < \bar{\bar{x}}_e(t) < R$ for $t \in [0, t^*)$. Thus,

$$\Psi_3(\bar{\bar{x}}_e) \leq k_3^R |\bar{\bar{x}}_e| , \ \forall \bar{\bar{x}}_e < R ,$$

with the positive constant k_3^R possibly dependent on R. Thus, $\bar{\bar{x}}_e$ satisfies

$$\dot{\bar{\bar{x}}}_e \leq (k_3^R - \lambda_1) \bar{\bar{x}}_e + k_2 , \ \bar{\bar{x}}_e(0) := c_2 |X_e(0))| , \quad (71)$$

which leads to

$$\bar{\bar{x}}_e \leq \frac{k_2}{(k_3^R - \lambda_1)} (e^{(k_3^R - \lambda_1)t} - 1) + c_2 |X_e(0)| e^{(k_3^R - \lambda_1)t} , \quad (72)$$

and, consequently, from (68), one has

$$|X_e(t)| \leq c_2 e^{\lambda_2 t} |X_e(0)| + k_4 (e^{\lambda_2 t} - 1) , \ \forall t \in [0, t^*) , \quad (73)$$

where $\lambda_2 := k_3^R - \lambda_1$. Since $t_e(\tau)$ is bounded by some class $\mathcal{K}$ function of τ, thus there exists $0 < \tau_1 \leq 1$ such that $t_e(\tau) \leq t^* < t_M$, $\forall \tau \in (0, \tau_1]$. Now, one can obtain the following norm bound for X_e,

$$|X_e(t)| \leq (k_5 + k_6 \tau) |X_e(0)| + \Psi_4(\tau) , \quad (74)$$

$\forall \tau \in (0, \tau_1]$, $\forall t \in [0, t_e(\tau)] \subset [0, t^*]$. Finally, from (74), $\|(X_e)_t\| \leq (k_5 + k_6 \tau) |X_e(0)| + \Psi_4(\tau)$, $\forall t \in [0, t_e]$, in addition, recalling that $e_0 = h_c^T X_e$, from (30), follows the proposition result (47), i.e.,

$$|z(t)| \quad \leq \quad k_7 |z(0)| + \Psi_5(\tau) . \quad (75)$$

■

B. Proof of Theorem 2

Given $R > 0$ and $0 < R_0 < R$, then for some $t^* \in (0, t_M)$ and $|z(0)| < R_0$ one has $|z(t)| < R$ for $t \in [0, t^*)$. Assume $t \in [0, t^*)$.

From (54), one has

$$\|(z)_t\| \leq k_{z1} |z(0)| + k_{z2} \mathcal{V}(\tau) + k_{z3} \|(\beta_{\mathcal{U}})_t\| , \quad (76)$$

and, from (55), follows that

$$\|(z)_t\| \leq \bar{k}_{z1} |z(0)| + \bar{k}_{z2} \mathcal{V}(\tau) + \tau \bar{k}_{z3} , \quad (77)$$

which is valid if $\tau < \frac{1}{k_{z3} k_{ue}}$, where $\bar{k}_{z1} := \frac{k_{z1}}{1 - \tau k_{z3} k_{ue}}$, $\bar{k}_{z2} := \frac{k_{z2}}{1 - \tau k_{z3} k_{ue}}$ and $\bar{k}_{z3} := \frac{k_{z3} k_{red}}{1 - \tau k_{z3} k_{ue}}$. Then, from (55)

$$\begin{aligned} \|(\beta_{\mathcal{U}})_t\| &\leq \tau k_{ue} \bar{k}_{z1} |z(0)| + \tau k_{ue} \bar{k}_{z2} \mathcal{V}(\tau) + \\ &+ \tau^2 k_{ue} \bar{k}_{z3} + \tau k_{red} , \\ &\leq \tau k_{ue} \bar{k}_{z1} |z(0)| + \tau k_{o1} , \end{aligned} \quad (78)$$

where $k_{o1} := k_{ue} \bar{k}_{z2} \mathcal{V}(\tau) + \tau k_{ue} \bar{k}_{z3} + k_{red}$. Thus, from (54), the following upper bound holds

$$\begin{aligned} |z(t)| &\leq k_{z1} |z(0)| e^{-\lambda_2 t} + k_{z2} \mathcal{V}(\tau) e^{-\lambda_2 t} + \\ &+ \tau k_{z4} |z(0)| + \tau k_{z5} , \end{aligned} \quad (79)$$

where $k_{z4} := k_{z3} k_{ue} \bar{k}_{z1}$ and $k_{z5} := k_{o1} k_{z3}$.

Rewrite (75) as

$$\begin{aligned} |z(t)| &\leq \left[k_{z1} e^{-\lambda_2 t} + \tau k_{z4} \right] |z(0)| + \\ &+ k_{z2} \mathcal{V}(\tau) e^{-\lambda_2 t} + \tau k_{z5} . \end{aligned} \quad (80)$$

Noting that, for $\tau < 1/k_{z4}$, there exists $T_1 > 0$ such that

$$\lambda_z := \left[k_{z1} e^{-\lambda_2 T_1} + \tau k_{z4} \right] \leq 1 ,$$

thus, for $i = 0, 1, \ldots$, one has

$$\begin{aligned} |z(\bar{t}_i + T_1)| &\leq \lambda_z |z(\bar{t}_i)| + \\ &+ k_{z2} \mathcal{V}(\tau) e^{-\lambda_2 T_1} + \tau k_{z5} . \end{aligned} \quad (81)$$

Then, the simple linear recursive inequality (81) holds and easily lead to the conclusion that, for τ small enough, the error system is semi-globally exponentially stable with respect to a residual set of order τ. Moreover, one can also conclude that the initial time is irrelevant in this analysis, thus the stability result holds $\forall t \geq \bar{t}_0 \geq 0$. ■

C. Proof of Proposition 3

In what follows, k_i denote positive constants that depends only on the plant-controller parameters and $\Psi_i(\cdot)$ denote functions of class $\mathcal{K}_\infty$. From Proposition 2, $|\varepsilon_0(t)| \leq |\varphi_m(t)|$ ($\forall t \in [t_1, t_M)$), consequently from (58)-(59), one has

$$|\varepsilon_0(t)| \leq |\varepsilon_0(t_k)| + a(k) + 3 \|(\bar{\beta}_{\mathcal{U}})_t\| , \quad (82)$$

which is valid $\forall t \in [t_1, t_M)$, where t_k is the greatest switching time such that $0 \leq t_k \leq t \leq t_{k+1} < t_M$ (note that t_k depends on t). The relation $|\bar{\beta}_{\mathcal{U}}(t_k)| \leq \|(\bar{\beta}_{\mathcal{U}})_t\|$ was used to derive (82). Now, considering the sequence $t_1, t_2, \ldots, t_k < t_M$ of switching time instants that belongs to the maximum time interval $[0, t_M)$ of definition of ε_0, the following recursive inequality is verified

$$|\varepsilon_0(t_{k+1})| \leq |\varepsilon_0(t_k)| + a(k) + 3 \|(\bar{\beta}_{\mathcal{U}})_{t_{k+1}}\| , \ (\forall k \geq 1) , \quad (83)$$

which leads to the conclusion that

$$|\varepsilon_0(t_k)| \leq |\varepsilon_0(t_1)| + \sum_{i=1}^{(k-1)} a(i) + 3 \sum_{i=1}^{(k-1)} \|(\bar{\beta}_{\mathcal{U}})_{t_{i+1}}\| , \ (\forall k \geq 2) . \quad (84)$$

Noting that $\|(\bar{\beta}_{\mathcal{U}})_{t_i}\| \leq \|(\bar{\beta}_{\mathcal{U}})_{t_{i+1}}\| (\forall i)$, a simple but more conservative upper bound is obtained from (84), resulting in

$$|\varepsilon_0(t_k)| \leq |\varepsilon_0(t_1)| + a_\Sigma(k) + 3(k-1) \|(\bar{\beta}_{\mathcal{U}})_{t_k}\| , \ (\forall k \geq 1) , \quad (85)$$

where

$$a_\Sigma(k) := \begin{cases} \sum_{i=1}^{(k-1)} a(i) , & k \geq 2 , \\ 0 , & k = 1 . \end{cases}$$

Now, from (85) and (82), one has

$$\begin{aligned}|\varepsilon_0(t)| &\leq |\varepsilon_0(t_1)| + a_\Sigma(k) + 3(k-1)\|(\bar{\beta}_\mathcal{U})_{t_k}\| + \\ &+ a(k) + 3\|(\bar{\beta}_\mathcal{U})_t\|\,,\ (\forall t \in [t_1, t_M),\ k \geq 1) \qquad (86)\end{aligned}$$

Noting that $\|(\bar{\beta}_\mathcal{U})_{t_k}\| \leq \|(\bar{\beta}_\mathcal{U})_t\|$ (since $t \geq t_k$), and redefining $a_\Sigma(k)$ by

$$a_\Sigma(k) := \sum_{i=1}^{k} a(i)\,,\ k \geq 1\,,$$

the following inequality holds $\forall t \in [t_1, t_M)$ (which implies $k \geq 1$)

$$|\varepsilon_0(t)| \leq |\varepsilon_0(t_1)| + a_\Sigma(k) + 3k\|(\bar{\beta}_\mathcal{U})_t\|\,. \qquad (87)$$

Now, since k, $a_\Sigma(k)$ and $\|(\bar{\beta}_\mathcal{U})_t\|$ are positive and increase as t increases, then inequality (87) is also verified in terms of the $\mathcal{L}_{\infty e}$ norm of ε_0 ($\forall t \in [t_1, t_M)$), i.e,

$$\|(\varepsilon_0)_{t,t_1}\| \leq |\varepsilon_0(t_1)| + a_\Sigma(k) + 3k\|(\bar{\beta}_\mathcal{U})_t\|\,. \qquad (88)$$

Note that the relation $\|(\bar{\beta}_\mathcal{U})_{t,t_1}\| \leq \|(\bar{\beta}_\mathcal{U})_t\|$ was used to derive inequality (88).

Moreover, from (31) and (33), one has

$$\bar{e}_0 = \varepsilon_0 - \beta_\mathcal{U} - e_F^0\,,$$

thus, the following two relation can be obtained

$$|\varepsilon_0(t_1)| \leq k_1|X_e(t_1)| + |\beta_\mathcal{U}(t_1)| + |e_F^0(t_1)|\,, \qquad (89)$$

$$\|(\bar{e}_0)_{t,t_1}\| \leq \|(\varepsilon_0)_{t,t_1}\| + \|(\beta_\mathcal{U})_{t,t_1}\| + \|(e_F^0)_{t,t_1}\|\,, \qquad (90)$$

where $k_1 := |h_L|$ and the relation $\bar{e}_0 = h_L^T X_e$ was used to derive (89), see (24) for details. From (43), one has

$$|e_F^0(t)| \leq R_1\,, \forall t \geq t_e\,,$$

where R_1 is given by

$$R_1 := (|x_f(0)| + (\bar{\bar{k}}_e + 1)|X_e(0)|)\,,$$

according to (44). Since $t_1 := \bar{t}_e \geq t_e$, then $|e_F^0(t_1)| \leq R_1$ and $\|(e_F^0)_{t,t_1}\| \leq R_1$. Recalling that $|\beta_\mathcal{U}(t_1)| \leq \|(\beta_\mathcal{U})_{t,t_1}\|$, from (89), (90) and (88), the following upper bound holds

$$\begin{aligned}\|(\bar{e}_0)_{t,t_1}\| &\leq k_1|X_e(t_1)| + 2R_1 + a_\Sigma(k) + \\ &+ 2\|(\beta_\mathcal{U})_{t,t_1}\| + 3k\|(\bar{\beta}_\mathcal{U})_t\|\,. \qquad (91)\end{aligned}$$

Note that, transforming the state realization (A_c, b_c, h_L^T) of the transfer function $ML(s)$, described in (4) with output $\bar{e}_0$ given by (24), into the regular form, the complete state X_e can be bounded by

$$\|(X_e)_{t,t_1}\| \leq k_2|X_e(t_1)| + k_3\|(\bar{e}_0)_{t,t_1}\|\,. \qquad (92)$$

The upper bound (92) was obtained from the solution of the state equation (in the regular form), with t_1 being the initial time. Thus, from (90) and (91), one has

$$\begin{aligned}\|(X_e)_{t,t_1}\| &\leq k_4|X_e(t_1)| + 2k_3R_1 + k_3a_\Sigma(k) + \\ &+ 2k_3\|(\beta_\mathcal{U})_{t,t_1}\| + 3k_3k\|(\bar{\beta}_\mathcal{U})_t\|\,, \qquad (93)\end{aligned}$$

where $k_4 := k_1k_3 + k_2$. Now, from Proposition 1, one has

$$|(X_e)_t| \leq k_5|X_e(0)| + \Psi_1(\tau)\,,\ \forall t \in [0, t_1]\,. \qquad (94)$$

Adding (93) and (94) one can conclude that, $\forall t \in [0, t_M)$

$$\begin{aligned}\|(X_e)_t\| &\leq k_6|z(0)| + k_3a_\Sigma(k) + \Psi_2(\tau) + \\ &+ 2k_3\|(\beta_\mathcal{U})_t\| + 3k_3k\|(\bar{\beta}_\mathcal{U})_t\|\,, \qquad (95)\end{aligned}$$

where the facts that $\|(\beta_\mathcal{U})_{t,t_1}\| \leq \|(\beta_\mathcal{U})_t\|$ and $|X_e(t_1)| \leq \|(X_e)_{t_1}\|$ were used. Note that the operators appearing in (34) and (57) are of order $\mathcal{O}(\tau)$. Then, recalling that $\bar{u}$ satisfies (11), $f(t)$ satisfies (15), and from Assumption (A7), one can conclude that

$$\|(\beta_\mathcal{U})_t\|\,,\ \|(\bar{\beta}_\mathcal{U})_t\| \leq \tau\Psi_3(\|(X_e)_t\|) + k_7\tau\,. \qquad (96)$$

Now, from the proof of Proposition 1 one can conclude that $\forall R_0 > 0$, there exist $\forall R \geq R_0$ and some $t^* \in (0, t_M)$, which is *independent of* τ (the time constant of the lead filter), such that $|X_e(t)| < R$ for $t \in [0, t^*)$. Thus,

$$\Psi_3(|X_e|) \leq k_8^R|X_e|\,,\ \forall|X_e| < R\,,$$

with the positive constant k_8^R possibly dependent on R. Now, from (95) and (96), one has

$$\begin{aligned}|X_e(t)| &\leq k_9|z(0)| + k_{10}a_\Sigma(k) + \\ &+ \Psi_4(\tau) + k_{11}k\tau + k_{12}\tau\,, \qquad (97)\end{aligned}$$

if $\tau \in (0, \tau_2]$, where $\tau_2 < \frac{1}{k_3k_8^R(2+3k)}$. Finally, recalling that $e_0 = h_c^T X_e$ then, from (30), follows the proposition result (62), i.e.,

$$\begin{aligned}|z(t)| &\leq k_{13}|z(0)| + k_{14}a_\Sigma(k) + \\ &+ \Psi_5(\tau) + k_{15}k\tau + k_{16}\tau\,. \qquad (98)\end{aligned}$$

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D. *Proof of Theorem 3*

The switchings of the monitoring function stop at some index k^*. Indeed, since $a(k)$ increases unboundedly as $k \to \infty$, there is a finite value of k_1 such that for $k \geq k_1$ one has $a(k) \geq (2R_1e^{\bar{\lambda}_2\bar{t}_e})$ (see (56)). Let k^* be the smallest such k_1. It is not difficult to conclude that k^* can be related to $R_0 := |z(0)|$, through R_1, given by (44). In fact one can write

$$k^* \leq \mathcal{V}_k(R_0) + k_0 \qquad (99)$$

where $k_0 > 0$ is a constant and $\mathcal{V}_0 \in \mathcal{K}$. Now, from Proposition 3, it follows that, for τ sufficiently small

$$|z(t)| \leq \mathcal{V}_z(R_0) + c_z \qquad (100)$$

where, $c_z > 0$ is a constant and $\mathcal{V}_z \in \mathcal{K}$. Thus, if $R > \mathcal{V}_z(R_0) + c_z$, the system will stay within the ball of radius R for all $t \geq 0$. Thus, stability with respect to the ball of radius c_z is therefore guaranteed for initial conditions in the R_0-ball. Since R_0 can be chosen arbitrarily large provided that τ is chosen sufficiently small, semi-global stability ensues . We can also conclude that, either the switching of the monitoring function φ_k stops at a correct sign or, if not, then the state z has to stay in a residual set of order τ for all future time after the last switching occurred. Indeed, suppose that the latter is not true. Then, by a reverse time dynamics of the case in which the correct control direction is used, for which a residual set of order τ was shown to be semi-globally attractive, the trajectories would have to diverge beyond the bound (100), which is absurd. Thus, in the (highly improbable) case of wrong final control direction decision, the system would have already converged to a residual set of order τ, independent of the radius R_0. If the control direction is correctly found at the last switching (at k^*), then Theorem 2 applies. This demonstrates the Theorem 3. ■